AD-A100 617 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS M--ETC(U)
MAY 81 P SACKS DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2214 NL

1 OF 1
AD A
100617

END
DATE
FILMED
7-81
DTIC

LEVEL

(2)

AD A100617

MRC Technical Summary Report #2214

EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS

Paul Sacks

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

May 1981

(Received March 2, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

LEVEL

(2)

AD A100517

MRC Technical Summary Report #2214

EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS.

Paul Sacks

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

May 1981

(Received March 2, 1981)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

81 6 23 072

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS

Paul Sacks

Technical Summary Report #2214
May 1981

ABSTRACT

A class of singular parabolic equations is studied. It is proved that an initial and boundary value problem associated with the equation has a bounded solution which is continuous in the interior of its domain of definition. The case of the inhomogeneous porous medium equation with bounded initial data is included.

AMS (MOS) Subject Classifications: 35K10, 35K15, 35K20, 35K55, 35K60, 35K65

Key Words: singular parabolic equation, porous medium equation, regularity

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS78-01245 and MCS80-02946.

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS

Paul Sacks

Technical Summary Report #2214
May 1981

ABSTRACT

A class of singular parabolic equations is studied. It is proved that an initial and boundary value problem associated with the equation has a bounded solution which is continuous in the interior of its domain of definition. The case of the inhomogeneous porous medium equation with bounded initial data is included.

AMS (MOS) Subject Classifications: 35K10, 35K15, 35K20, 35K55, 35K60, 35K65

Key Words: singular parabolic equation, porous medium equation, regularity

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS78-01245 and MCS80-02946.

SIGNIFICANCE AND EXPLANATION

In applications there arise nonlinear partial differential equations which change type at points where the solution takes on certain values. Examples are heat flow with a temperature dependent conductivity and the flow of a gas through a porous medium. Only in recent years has there been developed a fairly broad theory which permits one to prove existence of solutions to such equations. These solutions are defined in a generalized or weak sense, and it is not known, a priori, whether the solution has the derivatives appearing in the equation in a classical sense. In general, one is interested in knowing what smoothness or regularity properties the solution possesses.

In this paper we study the solutions to a certain class of such singular equations where the solution is required to satisfy given initial and boundary conditions. The principal result is that the solution is continuous at interior points of its domain of definition. As a by product of the techniques used here, we obtain a new proof of the existence of such solutions. In the case of unbounded space domains some of the existence results so obtained are new.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SIGNIFICANCE AND EXPLANATION

In applications there arise nonlinear partial differential equations which change type at points where the solution takes on certain values. Examples are heat flow with a temperature dependent conductivity and the flow of a gas through a porous medium. Only in recent years has there been developed a fairly broad theory which permits one to prove existence of solutions to such equations. These solutions are defined in a generalized or weak sense, and it is not known, a priori, whether the solution has the derivatives appearing in the equation in a classical sense. In general, one is interested in knowing what smoothness or regularity properties the solution possesses.

In this paper we study the solutions to a certain class of such singular equations where the solution is required to satisfy given initial and boundary conditions. The principal result is that the solution is continuous at interior points of its domain of definition. As a by product of the techniques used here, we obtain a new proof of the existence of such solutions. In the case of unbounded space domains some of the existence results so obtained are new.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EXISTENCE AND REGULARITY OF SOLUTIONS OF
INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS

Paul Sacks

SECTION 0.

In this paper we will discuss some regularity results for partial differential equations of degenerate parabolic type. The model equation to have in mind is the porous medium equation

$$v_t = \Delta(v^m) = \nabla\cdot(mv^{m-1}\nabla v) \qquad (x,t) \in \Omega \times (0,T) \tag{0.1}$$

For $m \neq 1$ the equation is singular or changes type at any point where the solution vanishes, hence the standard quasilinear regularity theory [14] may not be applied directly.

We will define a notion of weak solution for an initial and boundary value problem associated with an equation of the type 0.1 and then prove interior continuity of these solutions. There are basically two approaches to a problem of this sort; the first is to assume that one has a given weak solution, and then to derive estimates for this function directly from the definition of weak solution by choosing test functions in the right way. This method is used by DiBenedetto [11], [18] to prove the regularity of weak solutions of a wide class of equations, including 0.1, under some additional assumptions (e.g. boundedness) not required by the definition of solution.

The second approach, which is the one to be used here, is to regularize the problem in some way so that it falls under the classical quasilinear existence and regularity theories; these approximate solutions are shown to converge to a solution of the original

EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS

Paul Sacks

SECTION 0.

In this paper we will discuss some regularity results for partial differential equations of degenerate parabolic type. The model equation to have in mind is the porous medium equation

$$v_t = \Delta(v^m) = \nabla\cdot(mv^{m-1}\nabla v) \qquad (x,t) \in \Omega \times (0,T) \tag{0.1}$$

For $m \neq 1$ the equation is singular or changes type at any point where the solution vanishes, hence the standard quasilinear regularity theory [14] may not be applied directly.

We will define a notion of weak solution for an initial and boundary value problem associated with an equation of the type 0.1 and then prove interior continuity of these solutions. There are basically two approaches to a problem of this sort; the first is to assume that one has a given weak solution, and then to derive estimates for this function directly from the definition of weak solution by choosing test functions in the right way. This method is used by DiBenedetto [11], [18] to prove the regularity of weak solutions of a wide class of equations, including 0.1, under some additional assumptions (e.g. boundedness) not required by the definition of solution.

The second approach, which is the one to be used here, is to regularize the problem in some way so that it falls under the classical quasilinear existence and regularity theories; these approximate solutions are shown to converge to a solution of the original problem. A priori estimates are derived for the approximations which then remain valid for their limit function. Using this method we have an existence theorem as a by-product; if we have a uniqueness theorem, which is often the case, then we may say that the weak solution of the initial and boundary value problem is continuous.

Caffarelli and Friedman ([6], [7]) study the Cauchy problem for the equation 0.1 with $m > 1$ and *non-negative* initial data by replacing the initial data $v_0(x)$ by

Sponsored by the United States Army under Contract No. DAAG29-80-0041. This material is based upon work supported by the National Science Foundation under Grant Nos. MCS78-01245 and MCS80-02946.

$v_{0\varepsilon}(x) = v_0(x) + \varepsilon$. It follows from the maximum principle that the corresponding solutions $v_\varepsilon(x,t)$ satisfy $v_\varepsilon(x,t) > \varepsilon$ so that the equation never degenerates, and furthermore that the sequence $v_\varepsilon(x,t)$ converges monotonically downwards to a limit function $v(x,t)$. An estimate due to Aronson and Benilan [1], which is special to this problem, is then employed to obtain estimates on $v_\varepsilon(x,t)$ independently of ε.

In the paper [5], Caffarelli and Evans outline a more general method; in this case the equation itself is regularized with 0.1 being replaced by

$$v_t = \nabla\cdot((mv^{m-1} + \varepsilon)\nabla v) \ .$$

Specifically, they prove the following.

Let Ω be bounded in $\mathbf{R}^N$ with smooth boundary. Let $\phi(v)$ be a C^1 nondecreasing function on $\mathbf{R}$ which "looks like" $|v|^m \operatorname{sign} v$ for $m > 1$, (this will be made precise later) and suppose $\phi(v_0(x)) \in C_0^1(\bar{\Omega})$. Then the problem

$$\begin{aligned} v_t &= \Delta(\phi(v)) && (x,t) \in \Omega \times (0,T) \\ v(x,0) &= v_0(x) && x \in \Omega \\ v(x,t) &= 0 && x \in \partial\Omega \end{aligned}$$

has a unique solution $v(x,t) \in C(Q_T)$.

In this paper we prove analogous results for the inhomogeneous equation, and we consider also the case of unbounded domains.

In [15] corresponding results will be discussed for other type of nonlinearities, and different initial and boundary conditions. Regularity up to the boundary will also be discussed there.

The methods employed here are principally those of [5], combined with techniques which may be found, for example, in [14].

I would like to thank M. Pierre, E. DiBenedetto, and especially M. Crandall.

Notation:

$Q_T = \Omega \times (0,T) \qquad \Omega \subset \mathbf{R}^N$

$S_T = \partial\Omega \times [0,T)$

$\partial Q = S_T \cup (\Omega \times \{t = 0\})$ if Q is a cylinder in (x,t) space. (This is the parabolic boundary of Q). Otherwise this is the usual boundary.

$|A|$ = Lebesgue measure of the set A.

$\{v > k\}$ = set of points where $v > k$.

$Q_{x_0,t_0}(R) = \{(x,t) \in \mathbf{R}^{N+1} : |x - x_0| < R,\ t_0 - R^2 < t < t_0\}$. The subscript x_0,t_0 may be dropped if it is clear from the context.

$f^+ = \max(0,f)$.

$\nabla u, \Delta u$ = gradient and Laplacian in the space variables only.

$C(Q_T)$ = continuous functions on Q_T.

$D'(Q_T)$ = distributions on Q_T.

$W^{1,0}(Q_T) = \{u \in L^2(Q_T) : \nabla u \in L^2(Q_T)\}$

$W^{1,1}(Q_T) = \{u \in W^{1,0}(Q_T) : u_t \in L^2(Q_T)\}$.

$V_2(Q_T) = \{u : \sup\limits_{0<t<T} \|u(x,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)} < \infty\} = L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

A zero over any of these spaces indicates those functions which vanish on S_T. The summation convention is used, e.g. $(f_i)_{x_i} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}(x,t)$.

SECTION 1.

We consider the following initial and boundary value problem

$$\begin{aligned} [\beta(u)]_t &= \Delta u + F \qquad && (x,t) \in Q_T \\ u(x,0) &= u_0(x) && x \in \Omega \qquad\qquad (1.1) \\ u|_{S_T} &= 0 \end{aligned}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary.

We first describe the assumptions to be made on $u_0(x)$, F and β.

H1) $u_0(x) \in L^\infty(\Omega)$

$\partial Q = S_T \cup (\Omega \times \{t = 0\})$ if Q is a cylinder in (x,t) space. (This is the parabolic boundary of Q). Otherwise this is the usual boundary.

$|A|$ = Lebesgue measure of the set A.

$\{v > k\}$ = set of points where $v > k$.

$Q_{x_0,t_0}(R) = \{(x,t) \in \mathbf{R}^{N+1} : |x - x_0| < R,\ t_0 - R^2 < t < t_0\}$. The subscript x_0,t_0 may be dropped if it is clear from the context.

$f^+ = \max(0,f)$.

$\nabla u, \Delta u$ = gradient and Laplacian in the space variables only.

$C(Q_T)$ = continuous functions on Q_T.

$D'(Q_T)$ = distributions on Q_T.

$W^{1,0}(Q_T) = \{u \in L^2(Q_T) : \nabla u \in L^2(Q_T)\}$

$W^{1,1}(Q_T) = \{u \in W^{1,0}(Q_T) : u_t \in L^2(Q_T)\}$.

$V_2(Q_T) = \{u : \sup_{0<t<T} \|u(x,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)} < \infty\} = L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

A zero over any of these spaces indicates those functions which vanish on S_T. The summation convention is used, e.g. $(f_i)_{x_i} = \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}(x,t)$.

SECTION 1.

We consider the following initial and boundary value problem

$$\begin{aligned} [\beta(u)]_t &= \Delta u + F && (x,t) \in Q_T \\ u(x,0) &= u_0(x) && x \in \Omega \qquad (1.1) \\ u|_{S_T} &= 0 \end{aligned}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary.

We first describe the assumptions to be made on $u_0(x)$, F and β.

H1) $u_0(x) \in L^\infty(\Omega)$

For the purpose of stating the hypothesis on F we introduce some more notation.

$$G_p(Q_T) = \{F \in D'(Q_T) : F = f_0 + (f_i)_{x_i}, f_0, f_i^2 \in L^p(Q_T)\}$$

$$= L^p(Q_T) + L^{2p}((0,T);W^{-1,2p}(\Omega)) .$$

$\dot{G}_p$ is a Banach space with the norm

$$\|F\|_{G_p} = \inf\{\|f_0\|_{L^p} + \sum_{i=1}^{N} \|f_i\|_{L^{2p}} : F = f_0 + (f_i)_{x_i}\} .$$

Now we define a subset of G_p whose members have a certain approximation property.

$\hat{G}_p(Q_T) = \{F \in G_p(Q_T)$: there exists F_ε, $G_\varepsilon \in C^1(\bar{Q}_T)$ and $A < \infty$ satisfying

(i) $F_\varepsilon \to F$ in $G_p(Q_T)$ as $\varepsilon \to 0$

(ii) $\|F_\varepsilon\|_{G_p}$, $\|G_\varepsilon\|_{G_p} \le A$

(iii) For any $Q' \subset\subset Q_T$ there exists $\varepsilon_0 > 0$ depending only on $\mathrm{dist}(Q', \partial Q_T)$ such that $|F_\varepsilon| \le G_\varepsilon$ in Q' for $\varepsilon \le \varepsilon_0$.$\}$

Let us denote by $\langle F \rangle_p$ the smallest constant A which works in the above definition.

H2) $F \in \hat{G}_p(Q_T)$ for some $p > \frac{N+2}{2}$.

See the remarks following the statement of Theorem 1.1 for some discussion of this condition.

H3) We assume that β is locally absolutely continuous, $\beta(0) = 0$, and that there exist functions $\mu_1(\cdot), \mu_2(\cdot)$ such that

(i) $\beta'(s) \ge \mu_1(M) > 0$ for $s \in [-M,M]$

(ii) $\phi'(s) \ge \mu_2(\delta) > 0$ for $s \in [-\frac{1}{\delta}, -\delta] \cup [\delta, \frac{1}{\delta}]$, $\delta > 0$

where $\phi = \beta^{-1}$.

We will say that a function $u \in V_2(Q_T)$ is a solution of the initial and boundary value problem 1.1 if there exists $v \in L^1(Q_T)$, $v(x,t) = \beta(u(x,t))$ a.e. such that

$$\iint_{Q_T} (v\psi_t - \nabla u\cdot\nabla\psi + f_0\psi - f_i\psi_{x_i})dxdt + \int_\Omega \beta(u_0(x))\psi(x,0)dx = 0 \qquad (1.2)$$

for every $\psi \in E = \{\psi \in C^1(\bar{Q}_T) : \psi(x,t) = 0$ for $x \in \partial\Omega$ or $t = T\}$.

Theorem 1.1. Under assumptions H1, H2 and H3 the problem 1.1 has a solution $u \in L^\infty(Q_T) \cap C(Q_T)$. The solution is unique in the class of bounded functions. The norm $\|u\|_{L^\infty(Q_T)}$ depends only on

$$N,\ |\Omega|,\ T,\ \|u_0\|_{L^\infty},\ \|F\|_{G_p(Q_T)},\ p,\ \mu_1(\cdot),\ \mu_2(\cdot)$$

The modulus of continuity of u at a point separated from ∂Q_T by a distance d depends only on

$$N,\ \|u\|_{L^\infty(Q_T)},\ \langle F\rangle_p,\ p,\ \mu_1(\cdot),\ \mu_2(\cdot),\ d\ .$$

Remarks. (i) The condition H3 on β includes the case of the porous medium equation,

$$\beta(s) = |s|^{\frac{1}{m}}\text{signs} \quad m > 1\ .$$

(ii) In the case of quasilinear, non-degenerate, parabolic problems of the type 1.1, one obtains bounded continuous solutions for $F \in G_p(Q_T)$, $p > \frac{N+2}{2}$, [14]. We expect that this is true for 1.1 also, but the present proof requires the stronger assumption that $F \in \hat{G}_p(Q_T)$, $p > \frac{N+2}{2}$. G_p contains L^p and also G_p^+, the positive distributions belonging to G_p. To see this, let $F_\varepsilon = J_\varepsilon * F$ where J_ε is a standard mollifier. Whenever $F = f_0 + (f_i)_{x_i}$ we may define f_0, f_i to be zero outside Q_T, so that this definition makes sense. For the case $F \in L^p(Q_T)$, $F_\varepsilon \to F$ in L^p and $|F_\varepsilon| \le J_\varepsilon * |F| \equiv G_\varepsilon \to |F|$ in L^p by standard theory. In this case $\langle F\rangle_p = \|F\|_{L^p}$. For the case $F \in G_p^+(Q_T)$, $F_\varepsilon = (J_\varepsilon * f_0) + (J_\varepsilon * F_i)_{x_i} \equiv f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$. Again, $f_{0\varepsilon} \to f_0$ in L^p, $f_{i\varepsilon} \to f_i$ in L^{2p} so that $F_\varepsilon \to F$ in G_p. Also, for $\varepsilon < \text{dist}((x,t),\partial Q_T)$, $F_\varepsilon \ge 0$. We may therefore take $G_\varepsilon = F_\varepsilon$ and $\langle F\rangle_p = \|F\|_{G_p}$.

As an example, let $\Omega = (-1,1)$ in $\mathbf{R}^1$ and

$$f_1(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

$$\iint_{Q_T} (v\psi_t - \nabla u\cdot\nabla\psi + f_0\psi - f_i\psi_{x_i})dxdt + \int_\Omega \beta(u_0(x))\psi(x,0)dx = 0 \qquad (1.2)$$

for every $\psi \in E = \{\psi \in C^1(\bar{Q}_T) : \psi(x,t) = 0$ for $x \in \partial\Omega$ or $t = T\}$.

Theorem 1.1. Under assumptions H1, H2 and H3 the problem 1.1 has a solution $u \in L^\infty(Q_T) \cap C(Q_T)$. The solution is unique in the class of bounded functions. The norm $\|u\|_{L^\infty(Q_T)}$ depends only on

$$N,\ |\Omega|,\ T,\ \|u_0\|_{L^\infty},\ \|F\|_{G_p(Q_T)},\ p,\ \mu_1(\cdot),\ \mu_2(\cdot)$$

The modulus of continuity of u at a point separated from ∂Q_T by a distance d depends only on

$$N,\ \|u\|_{L^\infty(Q_T)},\ (F)_p,\ p,\ \mu_1(\cdot),\ \mu_2(\cdot),\ d\ .$$

Remarks. (i) The condition H3 on β includes the case of the porous medium equation,

$$\beta(s) = |s|^{\frac{1}{m}}\text{signs} \quad m > 1\ .$$

(ii) In the case of quasilinear, non-degenerate, parabolic problems of the type 1.1, one obtains bounded continuous solutions for $F \in G_p(Q_T)$, $p > \frac{N+2}{2}$, [14]. We expect that this is true for 1.1 also, but the present proof requires the stronger assumption that $F \in \hat{G}_p(Q_T)$, $p > \frac{N+2}{2}$. G_p contains L^p and also G_p^+, the positive distributions belonging to G_p. To see this, let $F_\varepsilon = J_\varepsilon * F$ where J_ε is a standard mollifier. Whenever $F = f_0 + (f_i)_{x_i}$ we may define f_0, f_i to be zero outside Q_T, so that this definition makes sense. For the case $F \in L^p(Q_T)$, $F_\varepsilon \to F$ in L^p and $|F_\varepsilon| \le J_\varepsilon * |F| \equiv G_\varepsilon \to |F|$ in L^p by standard theory. In this case $(F)_p = \|F\|_{L^p}$. For the case $F \in G_p^+(Q_T)$, $F_\varepsilon = (J_\varepsilon * f_0) + (J_\varepsilon * F_i)_{x_i} \equiv f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$. Again, $f_{0\varepsilon} \to f_0$ in L^p, $f_{i\varepsilon} \to f_i$ in L^{2p} so that $F_\varepsilon \to F$ in G_p. Also, for $\varepsilon < \text{dist}((x,t),\partial Q_T)$, $F_\varepsilon \ge 0$. We may therefore take $G_\varepsilon = F_\varepsilon$ and $(F)_p = \|F\|_{G_p}$.

As an example, let $\Omega = (-1,1)$ in $\mathbf{R}^1$ and

$$f_1(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

Then $F = (f_1)_x = \delta_0(x)$, a mass concentrated on the surface $\{x = 0,\ 0 < t < T\}$ which is a positive distribution belonging to $G_\infty(Q_T)$. More generally, if $f_i(x_i,\dots,x_N)$ is non-decreasing in x_i, then $F = (f_i)_{x_i}$ is a positive distribution.

Another type of sufficient condition that H2 be satisfied is the following. Suppose $F \in L^1(Q_T)$ and $|F| = (g_i)_{x_i}$ for some choice of $g_i \in L^{2p}$, $p > \frac{N+2}{2}$, that is $|F| \in L^{2p}((0,T);W^{-1,2p}(\Omega))$. It follows that F belongs to the same space, $F = (f_i)_{x_i}$ for some $f_i \in L^{2p}$, $p > \frac{N+2}{2}$. Extending by zero and mollifying as before we have

$$|F_\varepsilon| = |(f_{i\varepsilon})_{x_i}| \le J_\varepsilon * |F| = (g_{i\varepsilon})_{x_i} \equiv G_\varepsilon$$

for $\varepsilon < \operatorname{dist}((x,t),\partial Q_T)$. In this case $F_\varepsilon \to F$, $G_\varepsilon \to |F|$ in G_p and $\langle F\rangle_p = \| |F| \|_{G_p}$. More generally we could assume that $F \in L^1(Q_T)$ and $|F - f_0| = (g_i)_{x_i}$, $g_i \in L^{2p}$, for some $f_0 \in L^p$, $p > \frac{N+2}{2}$. As an example, let $\Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)$ in $\mathbf{R}^1$ and $F(x,t) = \dfrac{1}{x \log^2|x|}$. Then $|F| = \dfrac{\partial g}{\partial x}$ where $g(x,t) = \dfrac{-\operatorname{sign} x}{\log|x|}$. Here $g \in L^\infty(Q_T)$ and $F \in L^1(Q_T)$ but to no other L^p space.

(iii) For the proof of uniqueness see page 499 of [14] where the proof of the corresponding assertion for the Stefan problem is given. This method requires that $\phi = \beta^{-1}$ be locally Lipschitz and this follows from hypothesis H3. A uniqueness result under less restrictive conditions on ϕ may be found in [8].

(iv) Existence results for problem 1.1 are provided by nonlinear semigroup theory [2], [10]. The problem is rewritten as an abstract initial value problem

$$v_t + Av = f$$
$$v(0) = v_0$$

where $v : [0,T] \to X$, X is some Banach space of functions, A is a nonlinear operator in X, $f : [0,T] \to X$ and $v_0 \in X$. The boundary condition is incorporated into the definition of A. If this operator satisfies certain conditions then it may be shown that this problem has a solution $v \in C(0,T:X)$ if $f \in L^1(0,T:X)$ and $v_0 \in \overline{D(A)}$. In our case the operator A corresponds to the expression "$-\Delta(\phi(v))$"; good choices for X are $L^1(\Omega)$ and $H^{-1}(\Omega)$. The proofs that the corresponding operators satisfy the conditions of the abstract existence theorem may be found in [4] and [3] respectively. See also [9] for

Then $F = (f_1)_x = \delta_0(x)$, a mass concentrated on the surface $\{x = 0,\ 0 < t < T\}$ which is a positive distribution belonging to $G_\infty(Q_T)$. More generally, if $f_i(x_i, \dots, x_N)$ is non-decreasing in x_i, then $F = (f_i)_{x_i}$ is a positive distribution.

Another type of sufficient condition that H2 be satisfied is the following. Suppose $F \in L^1(Q_T)$ and $|F| = (g_i)_{x_i}$ for some choice of $g_i \in L^{2p}$, $p > \frac{N+2}{2}$, that is $|F| \in L^{2p}((0,T);W^{-1,2p}(\Omega))$. It follows that F belongs to the same space, $F = (f_i)_{x_i}$ for some $f_i \in L^{2p}$, $p > \frac{N+2}{2}$. Extending by zero and mollifying as before we have

$$|F_\varepsilon| = |(f_{i\varepsilon})_{x_i}| \le J_\varepsilon * |F| = (g_{i\varepsilon})_{x_i} \equiv G_\varepsilon$$

for $\varepsilon \le \mathrm{dist}((x,t), \partial Q_T)$. In this case $F_\varepsilon \to F$, $G_\varepsilon \to |F|$ in G_p and $\langle F \rangle_p = \| |F| \|_{G_p}$. More generally we could assume that $F \in L^1(Q_T)$ and $|F - f_0| = (g_i)_{x_i}$, $g_i \in L^{2p}$, for some $f_0 \in L^p$, $p > \frac{N+2}{2}$. As an example, let $\Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)$ in $\mathbf{R}^1$ and $F(x,t) = \dfrac{1}{x \log^2|x|}$. Then $|F| = \dfrac{\partial g}{\partial x}$ where $g(x,t) = \dfrac{-\mathrm{sign}\ x}{\log|x|}$. Here $g \in L^\infty(Q_T)$ and $F \in L^1(Q_T)$ but to no other L^p space.

(iii) For the proof of uniqueness see page 499 of [14] where the proof of the corresponding assertion for the Stefan problem is given. This method requires that $\phi = \beta^{-1}$ be locally Lipschitz and this follows from hypothesis H3. A uniqueness result under less restrictive conditions on ϕ may be found in [8].

(iv) Existence results for problem 1.1 are provided by nonlinear semigroup theory [2], [10]. The problem is rewritten as an abstract initial value problem

$$v_t + Av = f$$

$$v(0) = v_0$$

where $v : [0,T] \to X$, X is some Banach space of functions, A is a nonlinear operator in X, $f : [0,T] \to X$ and $v_0 \in X$. The boundary condition is incorporated into the definition of A. If this operator satisfies certain conditions then it may be shown that this problem has a solution $v \in C(0,T:X)$ if $f \in L^1(0,T:X)$ and $v_0 \in \overline{D(A)}$. In our case the operator A corresponds to the expression "$-\Delta(\phi(v))$"; good choices for X are $L^1(\Omega)$ and $H^{-1}(\Omega)$. The proofs that the corresponding operators satisfy the conditions of the abstract existence theorem may be found in [4] and [3] respectively. See also [9] for

the case $\Omega = \mathbf{R}^N$. If we assume that $f \in L^2(0,T : H^{-1}(\Omega))$ then it can be shown that there exists a solution $v \in C(0,T : H^{-1}(\Omega))$ which also satisfies $\frac{dv}{dt} \in L^2(0,T : H^{-1}(\Omega))$ and $u = \phi(v) \in L^2(0,T : H_0^1(\Omega))$; see [3]. Finally u also satisfies the equation 1.1 in the sense of distributions.

(v) The method of the proof of Theorem 1.1 is to obtain appropriate estimates on a sequence of functions which are then shown to converge to a solution of the integral equation 1.2. Setting $v = \beta(u)$, the equation 1.1 becomes formally

$$v_t = \Delta(\phi(v)) + F \tag{1.3}$$

We consider a regularized version of 1.3

$$\begin{aligned} v_t &= \Delta(\phi_n(v) + \varepsilon v) + F_\varepsilon(x,t) \\ v(x,0) &= v_{0\varepsilon}(x) \\ v|_{S_T} &= 0 \end{aligned} \qquad (1.4)\ (n,\varepsilon)$$

If ϕ_n, F_ε, $v_{0\varepsilon}$ are smooth enough, $\phi_n' > 0$, this problem has a classical solution ${}^n v^\varepsilon(x,t)$. The function ${}^n u^\varepsilon(x,t) = \phi_n({}^n v^\varepsilon(x,t))$ then satisfies

$$\begin{aligned} [\beta_n(u)]_t &= \Delta(u + \varepsilon\beta_n(u)) + F_\varepsilon \\ u(x,0) &= \phi_n(v_{0\varepsilon}(x)) \\ u|_{S_T} &= 0 \end{aligned} \qquad (1.5)\ (n,\varepsilon)$$

where $\beta_n = \phi_n^{-1}$. Therefore ${}^n u^\varepsilon$, ${}^n v^\varepsilon$ together satisfy

$$\iint_{Q_T} [v\psi_t - \nabla(u + \varepsilon v)\cdot\nabla\psi + F_\varepsilon\psi]\,dxdt + \int_\Omega v_{0\varepsilon}(x)\psi(x,0)dx = 0 \qquad (1.6)\ (n,\varepsilon)$$

for $\psi \in E$.

If we let $F_\varepsilon \to F$, $v_{0\varepsilon} \to v_0$, $\phi_n \to \phi$ we hope to obtain a limit function

$$u(x,t) = \lim_{\varepsilon\to 0} (\lim_{n\to\infty} {}^n u^\varepsilon(x,t))$$

which satisfies 1.1.

To prove such a result we must have various a priori bounds on the functions ${}^n u^\varepsilon$ and ${}^n v^\varepsilon$. It is relatively easy to obtain bounds in the spaces $L^\infty(Q_T)$ and $W^{1,1}(Q_T)$, and

the case $\Omega = \mathbf{R}^N$. If we assume that $f \in L^2(0,T : H^{-1}(\Omega))$ then it can be shown that there exists a solution $v \in C(0,T : H^{-1}(\Omega))$ which also satisfies $\frac{dv}{dt} \in L^2(0,T : H^{-1}(\Omega))$ and $u = \phi(v) \in L^2(0,T : H_0^1(\Omega))$; see [3]. Finally u also satisfies the equation 1.1 in the sense of distributions.

(v) The method of the proof of Theorem 1.1 is to obtain appropriate estimates on a sequence of functions which are then shown to converge to a solution of the integral equation 1.2. Setting $v = \beta(u)$, the equation 1.1 becomes formally

$$v_t = \Delta(\phi(v)) + F \tag{1.3}$$

We consider a regularized version of 1.3

$$\begin{aligned} v_t &= \Delta(\phi_n(v) + \varepsilon v) + F_\varepsilon(x,t) \qquad (1.4)\ (n,\varepsilon) \\ v(x,0) &= v_{0\varepsilon}(x) \\ v|_{S_T} &= 0 \end{aligned}$$

If ϕ_n, F_ε, $v_{0\varepsilon}$ are smooth enough, $\phi_n' > 0$, this problem has a classical solution ${}^n v^\varepsilon(x,t)$. The function ${}^n u^\varepsilon(x,t) = \phi_n({}^n v^\varepsilon(x,t))$ then satisfies

$$\begin{aligned} [\beta_n(u)]_t &= \Delta(u + \varepsilon\beta_n(u)) + F_\varepsilon \qquad (1.5)\ (n,\varepsilon) \\ u(x,0) &= \phi_n(v_{0\varepsilon}(x)) \\ u|_{S_T} &= 0 \end{aligned}$$

where $\beta_n = \phi_n^{-1}$. Therefore ${}^n u^\varepsilon$, ${}^n v^\varepsilon$ together satisfy

$$\iint_{Q_T} [v\psi_t - \nabla(u + \varepsilon v)\cdot\nabla\psi + F_\varepsilon\psi]\,dxdt + \int_\Omega v_{0\varepsilon}(x)\psi(x,0)dx = 0 \qquad (1.6)\ (n,\varepsilon)$$

for $\psi \in E$.

If we let $F_\varepsilon \to F$, $v_{0\varepsilon} \to v_0$, $\phi_n \to \phi$ we hope to obtain a limit function

$$u(x,t) = \lim_{\varepsilon\to 0}\,(\lim_{n\to\infty} {}^n u^\varepsilon(x,t))$$

which satisfies 1.1.

To prove such a result we must have various a priori bounds on the functions ${}^n u^\varepsilon$ and ${}^n v^\varepsilon$. It is relatively easy to obtain bounds in the spaces $L^\infty(Q_T)$ and $W^{1,1}(Q_T)$, and

this will be done in the next section. Furthermore, by a theorem due to Nash et. al., we may estimate the modulus of continuity of ${}^n u^\varepsilon$ and ${}^n v^\varepsilon$ for fixed $\varepsilon > 0$, independently of n. These bounds allow us to define $u^\varepsilon = \lim_{n\to\infty} {}^n u^\varepsilon$ and $v^\varepsilon = \lim_{n\to\infty} {}^n v^\varepsilon$ satisfying 1.6 with ϕ_n replaced by ϕ. In the remainder of the proof we will estimate the modulus of continuity of the functions u^ε. Attention will be focused first on a point $(x_0,t_0) \in Q_T$ where the solution vanishes. We construct a sequence of nested cylinders $Q(R_k)$ with top center (x_0,t_0) which shrink to the point (x_0,t_0), and a sequence $M_k \to 0$, such that $|u| \leq M_k$ on $Q(R_k)$. The sequences M_k and R_k do not depend on ε. This may then be used in conjuntion with the above mentioned theorem of Nash to prove the equicontinuity of the sequence u^ε.

In the concluding sections some extension of the theorem will be mentioned. 1) The res t remains true for certain unbounded domains. 2) The spaces L^p appearing in hypothesis H2 may be replaced by spaces

$$L^r((0,T) : L^q(\Omega))$$

for certain values of r and q.

SECTION 2.

The equation 1.4 (n,ε) may be written in divergence form.

$$v_t = \nabla\cdot((\phi_n'(v) + \varepsilon)\nabla v) + F_\varepsilon(x,t) \qquad (2.1)$$

If we let $F_\varepsilon \in C^1(\bar{Q}_T)$, $v_{0\varepsilon} \in C_0^\infty(\Omega)$ and $\phi_n \in C^\infty(\mathbb{R})$, $\phi_n' > 0$, then results from Chapter V, Section 6 of [14] guarantee the existence of a classical solution of 1.4 (n,ε). We now specify further properties of $v_{0\varepsilon}$, F_ε and ϕ_n.

(i) We have $v_0 \equiv \beta(u_0) \in L^\infty$ by the hypotheses. Pick $v_{0\varepsilon} \in C_0^\infty(\Omega)$ such that $v_{0\varepsilon} \to v_0$ in $L^2(\Omega)$. We may assume that $\|v_{0\varepsilon}\|_{L^\infty} \leq \|v_0\|_{L^\infty}$ and $\|v_{0\varepsilon}\|_{L^2} \leq \|v_0\|_{L^2}$.

(ii) Let $F_\varepsilon = f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$ where $f_{0\varepsilon}$ and $f_{i\varepsilon}$ are the functions assumed to exist in H2. Let $A = (F)_p$.

(iii) By mollification of ϕ we obtain a sequence $\phi_n \in C^\infty(\mathbb{R})$ satisfying $\phi_n(0) = 0$, $\phi_n' > 0$, and $\phi_n \to \phi$ uniformly on bounded sets of $\mathbb{R}$. Let $\beta_n = \phi_n^{-1}$. The conditions (i) and (ii) of H3 may be assumed to hold uniformly in n for

β_n and ϕ_n. β_n is a function and $\beta_n \to \beta$ pointwise and uniformly on any set $[-\frac{1}{\delta}, -\delta] \cup [\delta, \frac{1}{\delta}]$, $\delta > 0$. Note finally that pointwise bounds on β_n and ϕ_n may be derived from knowledge of $\mu_1(\cdot)$ and $\mu_2(\cdot)$ only.

The following will be considered the data of the problem 1.1

$$
\begin{array}{lll}
N, \ |\Omega|, \ T, \ \|u_0\|_{L^\infty(\Omega)} & & \\
p, \ \langle F \rangle_p & \text{from H2} & \qquad (2.2) \\
\mu_1(\cdot), \ \mu_2(\cdot) & \text{from H3} &
\end{array}
$$

First we state some lemmas which will be used in the proofs.

<u>Lemma 2.1</u>. There exists a constant C depending only on N such that

$$\|u\|_{L^{2\left(\frac{N+2}{N}\right)}(Q_T)} \leq C|u|_{V_2(Q_T)}$$

for $u \in \overset{\circ}{V}_2(Q_T)$. (See pages 74-75 of [14].)

<u>Lemma 2.2</u>. Let J_m be a sequence of nonnegative numbers satisfying

$$J_{m+1} \leq K_1(K_2)^m J_m^{1+b_1}$$

$$b_1 > 0, \quad K_1 > 0, \quad K_2 > 1.$$

Then $\lim\limits_{m\to\infty} J_m = 0$ provided $J_0 \leq k_1^{-\frac{1}{b_1}} k_2^{-\frac{1}{b_1^2}}$ (See page 95 of [14]).

<u>Lemma 2.3</u>. Suppose $\mu : [0,\infty) \to [0,\infty)$ is nonincreasing and there exist constants $C > 0$, $\alpha > 0$, $\beta > 1$ and $k_0 \geq 0$ such that

$$\mu(h) \leq \frac{C}{(h-k)^\alpha}[\mu(k)]^\beta \quad \text{for} \quad h > k > k_0$$

Then $\mu(d + k_0) = 0$ for $d = \left[C\mu(k_0)^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}\right]^{\frac{1}{\alpha}}$. (See [12] p. 63.)

It will now be shown that all solutions of problems 1.4 and 1.5 are bounded by some constant independently of n and ε. We will use the following notation: If $v(x,t)$ is

using Holder's inequality, so that

$$\|(v - k)^+\|^2_{L^{2(\frac{N+2}{N})}} \leq C\left(\int_0^T \int_{A_k(t)} f^2 dxdt + \left(\int_0^T \int_{A_k(t)} |f_0|^{\frac{2N+4}{N+4}} dxdt\right)^{\frac{N+4}{N+2}}\right) \qquad (2.6)$$

It follows that

$$\|(v - k)^+\|^2_{L^{2(\frac{N+2}{N})}} \leq C\left(\|f^2\|_p \mu(k)^{1 - \frac{1}{p}} + \|f_0\|_p^2 \mu(k)^{1 + \frac{2}{N+2} - \frac{2}{p}}\right) \qquad (2.7)$$

$$\leq C\left(\|f^2\|_p + \mu(k_0)^{\frac{2}{N+2} - \frac{1}{p}} \|f_0\|_p^2\right)\mu(k)^{1 - \frac{1}{p}} = C\mu(k)^{1 - \frac{1}{p}}$$

Now let $h > k > k_0$.

$$(h - k)^2 \mu(h)^{\frac{N}{N+2}} = (h - k)^2 \left(\int_0^T |A_h(t)| dt\right)^{\frac{N}{N+2}}$$

$$\leq \left(\int_0^T \int_{A_h(t)} (v - k)^{+2(\frac{N+2}{N})} dxdt\right)^{\frac{N}{N+2}}$$

$$\leq \left(\int_0^T \int_{A_k(t)} (v - k)^{+2(\frac{N+2}{N})} dxdt\right)^{\frac{N}{N+2}}$$

$$\leq C\mu(k)^{1 - \frac{1}{p}} \qquad \text{by (2.7)}$$

Therefore

$$\mu(h) \leq C^{\frac{N+2}{N}} \frac{\mu(k)^\beta}{(h - k)^\alpha} \qquad \begin{array}{l} \beta = \left(1 - \frac{1}{p}\right)\left(\frac{N+2}{N}\right) > 1 \\ \alpha = 2\left(\frac{N+2}{N}\right) \end{array}$$

$$\frac{1}{2}\int_0^{T^*}\left(\frac{d}{dt}\int_\Omega (v-k)^{+2}dx\right)dt + \int_0^{T^*}\int_{A_k(t)} a_{ij}v_{x_i}v_{x_j}dxdt \le$$

$$\int_0^{T^*}\int_{A_k(t)} |v_{x_i}||f_i|dxdt + \int_0^{T^*}\int_\Omega |f_0|(v-k)^+dxdt \le$$

$$\frac{\delta}{2}\int_0^{T^*}\int_{A_k(t)} |\nabla v|^2dxdt + C\int_0^{T^*}\int_{A_k(t)} f^2dxdt + \int_0^{T^*}\int_\Omega |f_0|(v-k)^+dxdt$$

where $f^2 = \sum_{i=1}^N f_i^2$.

This implies

$$\int_\Omega v(x,T^*) - k)^{+2}dx + \int_0^{T^*}\int_\Omega |\nabla(v-k)^+|^2dxdt \le$$

$$C\left(\int_0^{T^*}\int_{A_k(t)} f^2dxdt + \int_0^{T^*}\int_\Omega |f_0|(v-k)^+dxdt\right) .$$

Taking the supremum over $T^* \in [0,T]$ we have

$$|(v-k)^+|^2_{V_2(Q_T)} \le C\left(\int_0^T\int_{A_k(t)} f^2dxdt + \int_0^T\int_\Omega |f_0|(v-k)^+dxdt\right) \qquad (2.5)$$

Since $k > k_0$, $(v-k)^+ \in \overset{\circ}{V}_2(Q_T)$ so we may use Lemma 2.1.

$$\|(v-k)^+\|^2_{L^2\left(\frac{N+2}{2}\right)} \le C|(v-k)^+|^2_{V_2(Q_T)}$$

$$\le C\left(\int_0^T\int_{A_k(t)} f^2dxdt + \int_0^T\int_\Omega |f_0|(v-k)^+dxdt\right)$$

$$\le \frac{1}{2}\|(v-k)^+\|^2_{L^2\left(\frac{N+2}{N}\right)} + C\left(\int_0^T\int_{A_k(t)} f^2dxdt + \left(\int_0^T\int_{A_k(t)} |f_0|^{\frac{2N+4}{N+4}}dxdt\right)^{\frac{N+4}{N+2}}\right)$$

By Lemma 2.3

$$\mu(d + k_0) = 0 \quad \text{for} \quad d = \left(C^{\frac{N+2}{N}} \mu(k_0)^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}\right)^{\frac{1}{\alpha}} = C\mu(k_0)^{\frac{1}{N+2} - \frac{1}{2p}}$$

That is

$$v(x,t) \leq k_0 + C\mu(k_0)^{\frac{1}{N+2} - \frac{1}{2p}} \quad // \tag{2.8}$$

Remark. A similar estimate holds for $-v(x,t)$, depending on

$|\{(x,t) \in Q_T : v(x,t) \leq -k_0\}|$.

Corollary 2.1. There exist constants C_1 and C_2 depending only on the data 2.2, such that $|v(x,t)| \leq C_1$ and $|u(x,t)| \leq C_2$ where v is any solution of 1.4 and u is any solution of 1.5.

Proof. Choose $k_0 = \|v_0\|_{L^\infty}$ so that $v|_{\partial Q_T} \leq k_0$. We take $a_{ij}(x,t,v) = (\phi_n'(v) + \varepsilon)\delta_{ij}$ in the proposition, so that $a_{ij}(x,t,v)\xi_i\xi_j \geq \mu_2(k_0)|\xi|^2$ for $|v| \geq k_0$. Also $\mu(k_0) \leq |Q_T|$ so that $|v(x,t)| \leq C_1$ where C_1 depends only on quantities measured by the data 2.2. If u is a solution of 1.4 then $|u(x,t)| \leq \phi_n(C_1)$, and again this may be estimated by a constant C_2 which depends only on the data.

Corollary 2.2. Suppose the hypotheses of Proposition 2.1 hold with $\psi(x,t) \equiv 0$,

$$a_{ij}(x,t,v)\xi_i\xi_j \geq \delta|\xi|^2 \qquad \delta > 0$$

for all $x,t,v,$ and

$$Q_T = Q_{x_0,t_0}(R)$$

Then $|v(x,t)| \leq CR^{1 - \left(\frac{N+2}{2p}\right)}$ where C depends on N, p, $\|f_0\|_p, \|f_i^2\|_p$ and δ.

Proof. In the proposition we take $k_0 = 0$ and $\mu(0) = |Q(R)| = CR^{N+2}$. The conclusion then follows from 2.8.//

Remark. The above estimate remains valid if it is assumed only that $v \in V_2(Q_T) \cap C(\bar{Q}_T)$, and this will be used later. See p. 181 of [14].

We turn now to estimates of first order derivatives of solutions of 1.4 and 1.5. We assume that $v(x,t)$ is a solution of 1.4 (n,ε) and $u(x,t)$ is a solution of 1.5 (n,ε).

using Holder's inequality, so that

$$\|(v-k)^+\|^2_{L^{2(\frac{N+2}{N})}} \leq C\left(\int_0^T \int_{A_k(t)} f^2 dxdt + \left(\int_0^T \int_{A_k(t)} |f_0|^{\frac{2N+4}{N+4}} dxdt\right)^{\frac{N+4}{N+2}}\right) \qquad (2.6)$$

It follows that

$$\|(v-k)^+\|^2_{L^{2(\frac{N+2}{N})}} \leq C\left(\|f^2\|_p \mu(k)^{1-\frac{1}{p}} + \|f_0\|^2_p \mu(k)^{1+\frac{2}{N+2}-\frac{2}{p}}\right) \qquad (2.7)$$

$$\leq C\left(\|f^2\|_p + \mu(k_0)^{\frac{2}{N+2}-\frac{1}{p}}\|f_0\|^2_p\right)\mu(k)^{1-\frac{1}{p}} = C\mu(k)^{1-\frac{1}{p}}$$

Now let $h > k > k_0$.

$$(h-k)^2 \mu(h)^{\frac{N}{N+2}} = (h-k)^2\left(\int_0^T |A_h(t)|dt\right)^{\frac{N}{N+2}}$$

$$\leq \left(\int_0^T \int_{A_h(t)} (v-k)^{+2(\frac{N+2}{N})} dxdt\right)^{\frac{N}{N+2}}$$

$$\leq \left(\int_0^T \int_{A_k(t)} (v-k)^{+2(\frac{N+2}{N})} dxdt\right)^{\frac{N}{N+2}}$$

$$\leq C\mu(k)^{1-\frac{1}{p}} \qquad \text{by (2.7)}$$

Therefore

$$\mu(h) \leq C^{\frac{N+2}{N}} \frac{\mu(k)^\beta}{(h-k)^\alpha} \qquad \begin{array}{l} \beta = \left(1-\frac{1}{p}\right)\left(\frac{N+2}{N}\right) > 1 \\ \alpha = 2\left(\frac{N+2}{N}\right) \end{array}$$

Proposition 2.2.

(1) $\|\nabla v\|_{L^2(Q_T)} \le C(\varepsilon, \text{data})$

(2) $\|\nabla u\|_{L^2(Q_T)} \le C(\text{data})$

Proof. For the proof of (1), multiply 1.4 (n,ε) by v and integrate over Q_T.

$$\iint_{Q_T} \frac{\partial}{\partial t}\left(\frac{1}{2} v^2\right) dxdt + \iint_{Q_T} (\phi_n'(u) + \varepsilon)|\nabla v|^2 dxdt$$

$$\le \iint_{Q_T} f_{0\varepsilon} v \, dxdt - \iint f_{i\varepsilon} v_{x_i} dxdt .$$

Therefore

$$\varepsilon \iint_{Q_T} |\nabla v|^2 dxdt \le \frac{1}{2} \int_\Omega v_{0\varepsilon}^2 dx + \|f_{0\varepsilon}\|_{L^1} \|v\|_{L^\infty}$$

$$+ \frac{\varepsilon}{2} \iint_{Q_T} |\nabla v|^2 dxdt + \frac{1}{2\varepsilon} \iint_{Q_T} f_\varepsilon^2 dxdt$$

Thus

$$\varepsilon \|\nabla v\|_{L^2}^2 \le \|v_{0\varepsilon}\|_{L^2}^2 + 2\|f_{0\varepsilon}\|_{L^1} \|v\|_{L^\infty} + \frac{1}{\varepsilon} \|f_\varepsilon^2\|_{L^1} .$$

We have already estimated $\|v\|_{L^\infty}$ in terms of the data and $\|f_{0\varepsilon}\|_{L^1}$, $\|f_\varepsilon^2\|_{L^1}$, may be estimated by a constant depending on $(F)_p$ and $|\Omega|$. Therefore (1) is valid.

For the proof of (2) we multiply 1.5 (n,ε) by u and integrate over Q_T. Set $B(x) = \int_0^x s\beta_n'(s)ds$. We then get

$$\iint_{Q_T} |\nabla u|^2 dxdt \le 2 \int_\Omega B(\phi_n(v_{0\varepsilon}(x)))dx + 2\|f_{0\varepsilon}\|_{L^1} \|u\|_{L^\infty} + \|f_\varepsilon^2\|_{L^1}$$

Now $|B(x)| \le x\beta_n(x)$ so that

$$|B(\phi_n(v_{0\varepsilon}(x)))| \le |\phi_n(v_{0\varepsilon}(x))||v_{0\varepsilon}(x)| \le C|v_{0\varepsilon}(x)|^2$$

Thus the right hand side is bounded by a constant depending only on the data, as before.//

Remark. Similar estimates may be obtained for u_t and v_t by multiplying the equations 1.4 and 1.5 by $(u + \varepsilon v)_t$. Both estimates degenerate as $\varepsilon \to 0$ is general; however if

By Lemma 2.3

$$\mu(d + k_0) = 0 \quad \text{for} \quad d = \left(C^{\frac{N+2}{N}} \mu(k_0)^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}\right)^{\frac{1}{\alpha}} = C\mu(k_0)^{\frac{1}{N+2} - \frac{1}{2p}}$$

That is

$$v(x,t) \leq k_0 + C\mu(k_0)^{\frac{1}{N+2} - \frac{1}{2p}} \qquad // \tag{2.8}$$

Remark. A similar estimate holds for $-v(x,t)$, depending on $|\{(x,t) \in Q_T : v(x,t) \leq -k_0\}|$.

Corollary 2.1. There exist constants C_1 and C_2 depending only on the data 2.2, such that $|v(x,t)| \leq C_1$ and $|u(x,t)| \leq C_2$ where v is any solution of 1.4 and u is any solution of 1.5.

Proof. Choose $k_0 = \|v_0\|_{L^\infty}$ so that $v|_{\partial Q_T} \leq k_0$. We take $a_{ij}(x,t,v) = (\phi_n'(v) + \varepsilon)\delta_{ij}$ in the proposition, so that $a_{ij}(x,t,v)\xi_i\xi_j \geq \mu_2(k_0)|\xi|^2$ for $|v| \geq k_0$. Also $\mu(k_0) \leq |Q_T|$ so that $|v(x,t)| \leq C_1$ where C_1 depends only on quantities measured by the data 2.2. If u is a solution of 1.4 then $|u(x,t)| \leq \phi_n(C_1)$, and again this may be estimated by a constant C_2 which depends only on the data.

Corollary 2.2. Suppose the hypotheses of Proposition 2.1 hold with $\psi(x,t) \equiv 0$,

$$a_{ij}(x,t,v)\xi_i\xi_j \geq \delta|\xi|^2 \qquad \delta > 0$$

for all x,t,v, and

$$Q_T = Q_{x_0,t_0}(R)$$

Then $|v(x,t)| \leq CR^{1 - (\frac{N+2}{2p})}$ where C depends on N, p, $\|f_0\|_p, \|f_i^2\|_p$ and δ.

Proof. In the proposition we take $k_0 = 0$ and $\mu(0) = |Q(R)| = CR^{N+2}$. The conclusion then follows from 2.8.//

Remark. The above estimate remains valid if it is assumed only that $v \in V_2(Q_T) \cap C(\bar{Q}_T)$, and this will be used later. See p. 181 of [14].

We turn now to estimates of first order derivatives of solutions of 1.4 and 1.5. We assume that $v(x,t)$ is a solution of 1.4 (n,ε) and $u(x,t)$ is a solution of 1.5 (n,ε).

Proposition 2.2.

(1) $\|\nabla v\|_{L^2(Q_T)} \leq C(\varepsilon,\text{data})$

(2) $\|\nabla u\|_{L^2(Q_T)} \leq C(\text{data})$

Proof. For the proof of (1), multiply 1.4 (n,ε) by v and integrate over Q_T.

$$\iint_{Q_T} \frac{\partial}{\partial t}\left(\frac{1}{2} v^2\right)dxdt + \iint_{Q_T} (\phi_n'(u) + \varepsilon)|\nabla v|^2 dxdt$$

$$\leq \iint_{Q_T} f_{0\varepsilon} v \, dxdt - \iint f_{i\varepsilon} v_{x_i} dxdt .$$

Therefore

$$\varepsilon \iint_{Q_T} |\nabla v|^2 dxdt \leq \frac{1}{2}\int_\Omega v_{0\varepsilon}^2 dx + \|f_{0\varepsilon}\|_{L^1}\|v\|_{L^\infty}$$

$$+ \frac{\varepsilon}{2}\iint_{Q_T} |\nabla v|^2 dxdt + \frac{1}{2\varepsilon}\iint_{Q_T} f_\varepsilon^2 dxdt$$

Thus

$$\varepsilon\|\nabla v\|_{L^2}^2 \leq \|v_{0\varepsilon}\|_{L^2}^2 + 2\|f_{0\varepsilon}\|_{L^1}\|v\|_{L^\infty} + \frac{1}{\varepsilon}\|f_\varepsilon^2\|_{L^1} .$$

We have already estimated $\|v\|_{L^\infty}$ in terms of the data and $\|f_{0\varepsilon}\|_{L^1}$, $\|f_\varepsilon^2\|_{L^1}$, may be estimated by a constant depending on $(F)_p$ and $|\Omega|$. Therefore (1) is valid.

For the proof of (2) we multiply 1.5 (n,ε) by u and integrate over Q_T. Set $B(x) = \int_0^x s\beta_n'(s)ds$. We then get

$$\iint_{Q_T} |\nabla u|^2 dxdt \leq 2\int_\Omega B(\phi_n(v_{0\varepsilon}(x)))dx + 2\|f_{0\varepsilon}\|_{L^1}\|u\|_{L^\infty} + \|f_\varepsilon^2\|_{L^1}$$

Now $|B(x)| \leq x\beta_n(x)$ so that

$$|B(\phi_n(v_{0\varepsilon}(x)))| \leq |\phi_n(v_{0\varepsilon}(x))||v_{0\varepsilon}(x)| \leq C|v_{0\varepsilon}(x)|^2$$

Thus the right hand side is bounded by a constant depending only on the data, as before.//

Remark. Similar estimates may be obtained for u_t and v_t by multiplying the equations 1.4 and 1.5 by $(u + \varepsilon v)_t$. Both estimates degenerate as $\varepsilon \to 0$ is general; however if

$$\varepsilon|\xi|^2 \leq a_{ij}\xi_i\xi_j$$

$$\|a_{ij}\|_{L^\infty(Q_T)} \leq C(\text{data})$$

All quanties occurring in Theorem 2.1 for this case are given in terms of the data, d, and ε, thus 2.9 holds. Since $\{\phi_n\}$ are uniformly Lipschitz on bounded intervals 2.10 holds.//

Remark. Theorem 2.1 remains true if the equation contains lower order terms satisfying certain integrability conditions. Also, if the initial and boundary values are smooth enough the solution will be Holder continuous on $\bar{Q}_T$. For a precise statement see Chapter III, Section 10 of [14]. See also [12] pages 62-76 for the elliptic case.

Using the results of Corollary 2.1, Proposition 2.2 and Corollary 2.3 we may find a subsequence $n_k \to \infty$ and limit functions $u^\varepsilon(x,t)$ and $v^\varepsilon(x,t)$ so that ${}^{n_k}u^\varepsilon \to u^\varepsilon$, ${}^{n_k}v^\varepsilon \to v^\varepsilon$ uniformly on compact sets, $\nabla\, {}^{n_k}u^\varepsilon \to \nabla u^\varepsilon$, $\nabla\, {}^{n_k}v^\varepsilon \to \nabla v^\varepsilon$ weakly in $L^2(Q_T)$. Since $\phi_{n_k} \to \phi$ uniformly on bounded sets, $u^\varepsilon(x,t) = \phi(v^\varepsilon(x,t))$ in Q_T.

Now multiply equation 1.4 (n_k,ε) by $\psi \in E$ and replace $\phi_{n_k}(v)$ by u.

$$\iint_{Q_T} \left({}^{n_k}v^\varepsilon\psi_t - \nabla({}^{n_k}u^\varepsilon + \varepsilon\, {}^{n_k}v^\varepsilon)\cdot\nabla\psi + f_{0\varepsilon}\psi - f_{i\varepsilon}\psi_{x_i}\right)dxdt$$

$$+ \int_\Omega v_{0\varepsilon}(x)\psi(x,0)dx = 0$$

Now letting $n_k \to \infty$ we have

$$\iint_{Q_T} (v^\varepsilon\psi_t - \nabla(u^\varepsilon + \varepsilon v^\varepsilon)\cdot\nabla\psi + f_{0\varepsilon}\psi - f_{i\varepsilon}\psi_{x_i})dxdt$$
$$+ \int_\Omega v_{0\varepsilon}(x)\psi(x,0)dx = 0 . \tag{2.11}$$

The function $v^\varepsilon(x,t)$ satisfies

$F \in L^2(Q_T)$ and $u_0(x)$ is smooth enough then actually $\|u_t\|_{L^2(Q_T)} \leq C$ independently of ε.

It is also possible to estimate the modulus of continuity of solutions of the equation 1.4 (n,ε) for fixed $\varepsilon > 0$. This follows from the following important theorem which will also play a role later in the proof.

<u>Theorem 2.1</u>. *Let* $v(x,t)$ *be a solution of the linear equation*

$$v_t - (a_{ij}v_{x_j})_{x_i} = f_0 + (f_i)_{x_i} \qquad (x,t) \in Q_T$$

where $\delta|\xi|^2 \leq a_{ij}\xi_i\xi_j$, $\|a_{ij}\|_{L^\infty(Q_T)} \leq \frac{1}{\delta}$, $\delta > 0$.

$$f_0, f_i^2 \in L^p(Q_T) \qquad p > \frac{N+2}{2}$$

and let $\|v\|_{L^\infty(Q_T)} \leq C_1$. Suppose $Q' \subset\subset Q_T$ with $\text{dist}(Q', \partial Q_T) > d$.

Then v is Holder continuous in Q'.

The Holder exponent α depends only on N, δ and p, while the norm $\|v\|_{C^\alpha(Q')}$ depends only on N, C_1, δ, p, $\|f_0\|_p$, $\|f_i^2\|_p$ and d.

<u>Corollary 2.3</u>. Let $Q' \subset\subset Q_T$ with $\text{dist}(Q', \partial Q_T) > d$. Let v be the solution of 1.4 (n,ε) and u be the solution of 1.5 (n,ε). Then

$$\|v\|_{C^\alpha(Q')} \leq C(\varepsilon, d, \text{data}) \tag{2.9}$$

$$\|u\|_{C^\alpha(Q')} \leq C(\varepsilon, d, \text{data}) \tag{2.10}$$

The exponent α depends on ε and the data.

<u>Proof</u>. $v(x,t)$ satisfies the linear equation

$$v_t - (a_{ij}v_{x_j})_{x_i} = f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$$

with $a_{ij}(x,t) = (\phi_n'(v(x,t)) + \varepsilon)\delta_{ij}$, so that

$$v_t = \Delta(\phi(v) + \varepsilon v) + F_\varepsilon(x,t)$$

$$v(x,0) = v_{0\varepsilon}(x) \qquad (2.12)\ (\varepsilon)$$

$$v|_{S_T} = 0$$

SECTION 3.

Following [5] we now examine the behaviour of solutions near the points where they vanish. Ultimately, of course, we must show that the solution is small in a neighborhood of such a point; here it will first be shown that a kind of smallness property holds in an average sense. This is the main nonlinear ingredient in the proof of Theorem 1.1. The desired result follows as a corollary from the next proposition, which states that if a solution is close enough, on the average, to its maximum in some cylinder, then it is pointwise greater than half maximum on a smaller cylinder with the same vertex.

Proposition 3.1. Let u be a classical solution in $Q(R) \subset Q_T$ of the equation

$$[\beta(u)]_t = \Delta(u + \varepsilon\beta(u)) + f_0 + (f_i)_{x_i} \qquad (3.1)$$

where $f_0, f_i^2 \in L^p(Q_T)$ for $p > \frac{N+2}{2}$, β satisfies H3) and also $\beta \in C^1(\mathbb{R})$. Let $u \leq C_2$ in Q_∞ and

$$0 < \max_{Q(R)} u \leq M \leq C_2$$

Then there exist constants $\alpha_0 > 0$, $p_0 > 0$ depending only on

$$N,\ C_2,\ \mu_1(C_2),\ \mu_2(\cdot),\ p,\ \|f_0\|_p,\ \|f_i^2\|_p \qquad (3.2)$$

such that if

$$\frac{1}{|Q(R)|} \iint_{Q(R)} (M - u)\,dxdt \leq \alpha_0 M^{p_0}$$

then

$$u \geq \frac{M}{2} \quad \text{in} \quad Q\left(\frac{R}{2}\right) .$$

Proof. C will denote any constant depending only on N, C_2, $\mu_1(C_2)$, $\mu_2(\cdot)$. Recall that pointwise estimates of $\beta(s)$ depend only on the function $\mu_2(\cdot)$.

$$\varepsilon|\xi|^2 \le a_{ij}\xi_i\xi_j$$

$$\|a_{ij}\|_{L^\infty(Q_T)} \le C(\text{data})$$

All quanties occurring in Theorem 2.1 for this case are given in terms of the data, d, and ε, thus 2.9 holds. Since $\{\phi_n\}$ are uniformly Lipschitz on bounded intervals 2.10 holds.//

Remark. Theorem 2.1 remains true if the equation contains lower order terms satisfying certain integrability conditions. Also, if the initial and boundary values are smooth enough the solution will be Holder continuous on $\bar{Q}_T$. For a precise statement see Chapter III, Section 10 of [14]. See also [12] pages 62-76 for the elliptic case.

Using the results of Corollary 2.1, Proposition 2.2 and Corollary 2.3 we may find a subsequence $n_k \to \infty$ and limit functions $u^\varepsilon(x,t)$ and $v^\varepsilon(x,t)$ so that ${}^{n_k}u^\varepsilon \to u^\varepsilon$, ${}^{n_k}v^\varepsilon \to v^\varepsilon$ uniformly on compact sets, $\nabla\, {}^{n_k}u^\varepsilon \to \nabla u^\varepsilon$, $\nabla\, {}^{n_k}v^\varepsilon \to \nabla v^\varepsilon$ weakly in $L^2(Q_T)$. Since $\phi_{n_k} \to \phi$ uniformly on bounded sets, $u^\varepsilon(x,t) = \phi(v^\varepsilon(x,t))$ in Q_T.

Now multiply equation 1.4 (n_k,ε) by $\psi \in E$ and replace $\phi_{n_k}(v)$ by u.

$$\iint_{Q_T} \left({}^{n_k}v^\varepsilon\psi_t - \nabla({}^{n_k}u^\varepsilon + \varepsilon\, {}^{n_k}v^\varepsilon)\cdot\nabla\psi + f_{0\varepsilon}\psi - f_{i\varepsilon}\psi_{x_i}\right)dxdt$$
$$+ \int_\Omega v_{0\varepsilon}(x)\psi(x,0)dx = 0$$

Now letting $n_k \to \infty$ we have

$$\iint_{Q_T} (v^\varepsilon\psi_t - \nabla(u^\varepsilon + \varepsilon v^\varepsilon)\cdot\nabla\psi + f_{0\varepsilon}\psi - f_{i\varepsilon}\psi_{x_i})dxdt$$
$$+ \int_\Omega v_{0\varepsilon}(x)\psi(x,0)dx = 0 \,. \tag{2.11}$$

The function $v^\varepsilon(x,t)$ satisfies

Set $w = M - u$ so that

$$\beta'(M - w)w_t = \Delta(w - \varepsilon\beta(M - w)) - f_0 - (f_i)_{x_i} \tag{3.3}$$

Let $\zeta(x,t)$ be a smooth test function, $0 \leq \zeta \leq 1$ and $\zeta = 0$ near $\partial Q(R)$. Let $0 \leq k \leq M$. We multiply 3.3 by $(w - k)^+\zeta^2$ and integrate over $B(x_0,R) \times (t_0 - R^2, t_0 - R^2 + t^*)$ for $t^* \in [0,R^2]$.

$$\begin{aligned}\iint \beta'(M - w)w_t(w - k)^+\zeta^2 dxdt &= - \iint \nabla(w - \varepsilon\beta(M - w))\cdot\nabla((w - k)^+\zeta^2)dxdt \\ &\quad - \iint f_0(w - k)^+\zeta^2 dxdt \\ &\quad + \iint f_i((w - k)^+\zeta^2)_{x_i} dxdt \\ &= I + II + III\end{aligned}$$

$$\begin{aligned}I &= - \iint |\nabla(w - k)^+|^2\zeta^2 dxdt - \varepsilon \iint \beta'(M - w)|\nabla(w - k)^+|^2\zeta^2 dxdt \\ &\quad + 2\varepsilon \iint (\nabla\beta(M - w)\cdot\nabla\zeta)\zeta(w - k)^+ dxdt - 2 \iint (\nabla(w - k)^+\cdot\nabla\zeta)\zeta(w - k)^+ dxdt \\ &\leq - \frac{3}{4} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt + C \iint (w - k)^{+2}|\nabla\zeta|^2 dxdt \\ &\quad + 2\varepsilon \iint (\nabla\beta(M - w)\cdot\nabla\zeta)\zeta(w - k)^+ dxdt\end{aligned}$$

$$\begin{aligned}III &= \iint \zeta^2 f_i(w - k)^+_{x_i} dxdt + 2 \iint \zeta(w - k)^+ f_i\zeta_{x_i} dxdt \\ &\leq \frac{1}{4} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt + \iint (w - k)^{+2}|\nabla\zeta|^2 dxdt \\ &\quad + 2 \iint_{\{w>k\}} f^2\zeta^2 dxdt \qquad f^2 \equiv \sum_{i=1}^N f_i^2\end{aligned}$$

Thus

$$\begin{aligned}&\iint \beta'(M - w)w_t(w - k)^+\zeta^2 dxdt + \frac{1}{2} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt \\ &\quad \leq C \iint (w - k)^{+2}|\nabla\zeta|^2 dxdt + \iint |f_0|\zeta^2(w - k)^+ dxdt \\ &\quad + 2 \iint_{\{w>k\}} f^2\zeta^2 dxdt + 2\varepsilon \iint (\nabla\beta(M - w)\cdot\nabla\zeta)\zeta(w - k)^+ dxdt\end{aligned} \tag{3.4}$$

To estimate the last term on the right in 3.4 we integrate by parts once more.

$$\begin{aligned}&- 2\varepsilon \iint \beta(M - w)\zeta\nabla\zeta\cdot\nabla(w - k)^+ dxdt - 2\varepsilon \iint \beta(M - w)(w - k)^+\nabla\cdot(\zeta\nabla\zeta)dxdt \\ &\quad \leq \frac{1}{4} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt + C \iint_{\{w>k\}\cap\operatorname{supp}\zeta} (|\nabla\zeta|^2 + |\Delta\zeta|)dxdt\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
v_t &= \Delta(\phi(v) + \varepsilon v) + F_\varepsilon(x,t) \\
v(x,0) &= v_{0\varepsilon}(x) \qquad\qquad (2.12)\ (\varepsilon) \\
v|_{S_T} &= 0
\end{aligned}
$$

SECTION 3.

Following [5] we now examine the behaviour of solutions near the points where they vanish. Ultimately, of course, we must show that the solution is small in a neighborhood of such a point; here it will first be shown that a kind of smallness property holds in an average sense. This is the main nonlinear ingredient in the proof of Theorem 1.1. The desired result follows as a corollary from the next proposition, which states that if a solution is close enough, on the average, to its maximum in some cylinder, then it is pointwise greater than half maximum on a smaller cylinder with the same vertex.

Proposition 3.1. Let u be a classical solution in $Q(R) \subset Q_T$ of the equation

$$[\beta(u)]_t = \Delta(u + \varepsilon\beta(u)) + f_0 + (f_i)_{x_i} \qquad (3.1)$$

where $f_0, f_i^2 \in L^p(Q_T)$ for $p > \frac{N+2}{2}$, β satisfies H3) and also $\beta \in C^1(\mathbb{R})$. Let $u \leq C_2$ in Q_∞ and

$$0 < \max_{Q(R)} u \leq M \leq C_2$$

Then there exist constants $\alpha_0 > 0$, $p_0 > 0$ depending only on

$$N,\ C_2,\ \mu_1(C_2),\ \mu_2(\cdot),\ p,\ \|f_0\|_p,\ \|f_i^2\|_p \qquad (3.2)$$

such that if

$$\frac{1}{|Q(R)|} \iint_{Q(R)} (M - u)\,dxdt \leq \alpha_0 M^{p_0}$$

then

$$u \geq \frac{M}{2} \quad \text{in} \quad Q(\frac{R}{2}) .$$

Proof. C will denote any constant depending only on N, C_2, $\mu_1(C_2)$, $\mu_2(\cdot)$. Recall that pointwise estimates of $\beta(s)$ depend only on the function $\mu_2(\cdot)$.

To estimate the first term on the left in 3.4, define $B(r) = \int_0^r \beta'(M - k - s)sds$, so that
$$\frac{d}{dt} B(w - k)^+ = \beta'(M - w)(w - k)^+ w_t$$
We have

$$B(w - k)^+ \geq \mu_1 \int_0^{(w-k)^+} sds = \frac{\mu_1}{2} (w - k)^{+2} \qquad \mu_1 = \mu_1(C_2)$$

$$B(w - k)^+ \leq (w - k)^+ \int_0^{(w-k)^+} \beta'(M - k - s)ds \leq C(w - k)^+$$

Therefore

$$\iint \beta'(M - w)(w - k)^+ \zeta^2 w_t dxdt = \int_{B(x_0,R)} B(w - k)^+ \zeta^2 \Big|_{t=t^*} dx$$

$$- 2 \iint B(w - k)^+ \zeta\zeta_t dxdt$$

$$\geq \frac{\mu_1}{2} \int_{B(x_0,R)} (w - k)^{+2} \zeta^2 \Big|_{t=t^*} dx - C \iint (w - k)^+ |\zeta\zeta_t| dxdt \qquad (3.6)$$

Combining 3.4, 3.5 and 3.6 we obtain

$$\int_{B(x_0,R)} (w - k)^+ \zeta^2 \Big|_{t=t^*} dx + \iint |\nabla(w - k)^+|^2 \zeta^2 dxdt$$

$$\leq C \iint_{\{w>k\}\cap \mathrm{supp}\,\zeta} (|\nabla\zeta|^2 + |\Delta\zeta| + |\zeta_t|) dxdt$$

$$+ C \iint |f_0|(w - k)^+ \zeta^2 dxdt + C \iint_{\{w>k\}\cap \mathrm{supp}\,\zeta} f^2 dxdt$$

Now, in the second term on the left the integrand may be replaced by $|\nabla(\zeta(w - k)^+)|^2$, since the error made may be estimated by terms of the form already appearing on the right side of 3.7. If we now take the supremum of the left side over $t^* \in [0,R^2]$ we obtain the estimate

Set $w = M - u$ so that

$$\beta'(M - w)w_t = \Delta(w - \varepsilon\beta(M - w)) - f_0 - (f_i)_{x_i} \tag{3.3}$$

Let $\zeta(x,t)$ be a smooth test function, $0 \le \zeta \le 1$ and $\zeta = 0$ near $\partial Q(R)$. Let $0 \le k \le M$. We multiply 3.3 by $(w - k)^+\zeta^2$ and integrate over $B(x_0,R) \times (t_0 - R^2, t_0 - R^2 + t^*)$ for $t^* \in [0,R^2]$.

$$\begin{aligned}\iint \beta'(M - w)w_t(w - k)^+\zeta^2 dxdt &= - \iint \nabla(w - \varepsilon\beta(M - w))\cdot\nabla((w - k)^+\zeta^2)dxdt \\ &\quad - \iint f_0(w - k)^+\zeta^2 dxdt \\ &\quad + \iint f_i((w - k)^+\zeta^2)_{x_i} dxdt \\ &= I + II + III\end{aligned}$$

$$\begin{aligned}I &= - \iint |\nabla(w - k)^+|^2\zeta^2 dxdt - \varepsilon \iint \beta'(M - w)|\nabla(w - k)^+|^2\zeta^2 dxdt \\ &\quad + 2\varepsilon \iint (\nabla\beta(M - w)\cdot\nabla\zeta)\zeta(w - k)^+ dxdt - 2 \iint (\nabla(w - k)^+\cdot\nabla\zeta)\zeta(w - k)^+ dxdt \\ &\le - \frac{3}{4} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt + C \iint (w - k)^{+2}|\nabla\zeta|^2 dxdt \\ &\quad + 2\varepsilon \iint (\nabla\beta(M - w)\cdot\nabla\zeta)\zeta(w - k)^+ dxdt\end{aligned}$$

$$\begin{aligned}III &= \iint \zeta^2 f_i(w - k)^+_{x_i} dxdt + 2 \iint \zeta(w - k)^+ f_i\zeta_{x_i} dxdt \\ &\le \frac{1}{4} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt + \iint (w - k)^{+2}|\nabla\zeta|^2 dxdt \\ &\quad + 2 \iint_{\{w>k\}} f^2\zeta^2 dxdt \qquad f^2 \equiv \sum_{i=1}^N f_i^2\end{aligned}$$

Thus

$$\begin{aligned}&\iint \beta'(M - w)w_t(w - k)^+\zeta^2 dxdt + \frac{1}{2} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt \\ &\quad \le C \iint (w - k)^{+2}|\nabla\zeta|^2 dxdt + \iint |f_0|\zeta^2(w - k)^+ dxdt \\ &\quad + 2 \iint_{\{w>k\}} f^2\zeta^2 dxdt + 2\varepsilon \iint (\nabla\beta(M - w)\cdot\nabla\zeta)\zeta(w - k)^+ dxdt\end{aligned} \tag{3.4}$$

To estimate the last term on the right in 3.4 we integrate by parts once more.

$$- 2\varepsilon \iint \beta(M - w)\zeta\nabla\zeta\cdot\nabla(w - k)^+ dxdt - 2\varepsilon \iint \beta(M - w)(w - k)^+\nabla\cdot(\zeta\nabla\zeta)dxdt \tag{3.5}$$

$$\le \frac{1}{4} \iint |\nabla(w - k)^+|^2\zeta^2 dxdt + C \iint_{\{w>k\}\cap \operatorname{supp}\zeta} (|\nabla\zeta|^2 + |\Delta\zeta|)dxdt$$

$$\|\zeta(w-k)^+\|^2_{V_2(Q(R))} \le C \iint_{\{w>k\}\cap\operatorname{supp}\zeta} (|\nabla\zeta|^2 + |\Delta\zeta| + |\zeta_t| + f^2)\,dxdt$$

$$+ C \iint_{Q(R)} |f_0|(w-k)^+\zeta^2\,dxdt \qquad (3.8)$$

Since $\zeta(w-k)^+ \in \overset{\circ}{V}_2(Q(R))$ we may use Lemma 2.1 to get

$$\|\zeta(w-k)^+\|^2_{L^{2\left(\frac{N+2}{N}\right)}(Q(R))} \le C \iint_{\{w>k\}\cap\operatorname{supp}\zeta} (|\nabla\zeta|^2 + |\Delta\zeta| + |\zeta_t| + f^2)\,dxdt$$

$$+ C\Big(\iint_{\{w>k\}\cap\operatorname{supp}\zeta} |f_0|^{\frac{2N+4}{N+4}}dxdt\Big)^{\frac{N+4}{N+2}} \qquad (3.9)$$

This inequality will now be iterated on a sequence of shrinking cylinders. Set $k_m = \frac{M}{2}\left(1 - \frac{1}{2^m}\right)$, $R_m = \frac{R}{2}\left(1 + \frac{1}{2^m}\right)$, $Q_m = Q(R_m)$. Choose smooth test functions $\zeta_m(x,t)$ such that $0 \le \zeta_m \le 1$, $\zeta_m \equiv 1$ on Q_{m+1}, $\zeta_m = 0$ near ∂Q_m, and $|\nabla\zeta_m|^2, |\Delta\zeta_m|, |\zeta_{m+1}| \le \frac{C4^m}{R^2}$. Define

$$J_m = \frac{1}{|Q_0|}\iint_{Q_m} (w-k_m)^{+2}dxdt\ .$$

We will show that $\{J_m\}$ satisfies the hypotheses of Lemma 2.2. It will follow that if $J_0 = \frac{1}{|Q(R)|}\iint_{Q(R)} (M-u)^2dxdt$ is small enough then $\iint_{Q\left(\frac{R}{2}\right)} \left(\frac{M}{2} - u\right)^{+2}dxdt = 0$ i.e. $u \ge \frac{M}{2}$ on $Q\left(\frac{R}{2}\right)$.

We have

$$|Q_0|J_{m+1} \le \Big(\iint_{Q_{m+1}} (w-k_{m+1})^{+2\left(\frac{N+2}{N}\right)}dxdt\Big)^{\frac{N}{N+2}}|Q_{m+1}\cap\{w>k_{m+1}\}|^{\frac{2}{N+2}}$$

$$\le \Big(\iint_{Q_{m+1}} (w-k_{m+1})^{+2\left(\frac{N+2}{N}\right)}dxdt\Big)^{\frac{N}{N+2}}|Q_m\cap\{w>k_{m+1}\}|^{\frac{2}{N+2}} \qquad (3.10)$$

To estimate the first term on the left in 3.4, define $B(r) = \int_0^r \beta'(M - k - s)sds$, so that

$$\frac{d}{dt} B(w - k)^+ = \beta'(M - w)(w - k)^+ w_t$$

We have

$$B(w - k)^+ \geq \mu_1 \int_0^{(w-k)^+} sds = \frac{\mu_1}{2} (w - k)^{+2} \qquad \mu_1 = \mu_1(C_2)$$

$$B(w - k)^+ \leq (w - k)^+ \int_0^{(w-k)^+} \beta'(M - k - s)ds \leq C(w - k)^+$$

Therefore

$$\iint \beta'(M - w)(w - k)^+ \zeta^2 w_t dxdt = \int_{B(x_0,R)} B(w - k)^+ \zeta^2 \Big|_{t=t^*} dx$$

$$- 2 \iint B(w - k)^+ \zeta\zeta_t dxdt$$

$$\geq \frac{\mu_1}{2} \int_{B(x_0,R)} (w - k)^{+2} \zeta^2 \Big|_{t=t^*} dx - C \iint (w - k)^+ |\zeta\zeta_t| dxdt \qquad (3.6)$$

Combining 3.4, 3.5 and 3.6 we obtain

$$\int_{B(x_0,R)} (w - k)^+ \zeta^2 \Big|_{t=t^*} dx + \iint |\nabla(w - k)^+|^2 \zeta^2 dxdt$$

$$\leq C \iint_{\{w>k\}\cap \operatorname{supp}\zeta} (|\nabla\zeta|^2 + |\Delta\zeta| + |\zeta_t|)dxdt$$

$$+ C \iint |f_0|(w - k)^+ \zeta^2 dxdt + C \iint_{\{w>k\}\cap \operatorname{supp}\zeta} f^2 dxdt$$

Now, in the second term on the left the integrand may be replaced by $|\nabla(\zeta(w - k)^+)|^2$, since the error made may be estimated by terms of the form already appearing on the right side of 3.7. If we now take the supremum of the left side over $t^* \in [0,R^2]$ we obtain the estimate

We now use the estimate 3.9 with $k = k_{m+1}$, $\zeta = \zeta_m$ and $Q(R) = Q_m$.

$$\Big(\iint_{Q_{m+1}} (w - k_{m+1})^{+2\left(\frac{N+2}{2}\right)} dxdt\Big)^{\frac{N}{N+2}} \leq \Big(\iint_{Q_m} \big((w - k_{m+1})^+ \zeta_m\big)^{2\left(\frac{N+2}{N}\right)} dxdt\Big)^{\frac{N}{N+2}}$$

$$\leq C \iint_{Q_m \cap \{w > k_{m+1}\}} (|\nabla \zeta_m|^2 + |\Delta \zeta_m| + |\zeta_{mt}| + f^2)\, dxdt$$

$$+ C\Big(\iint_{Q_m \cap \{w > k_{m+1}\}} |f_0|^{\frac{2N+4}{N+4}} dxdt\Big)^{\frac{N+4}{N+2}} = I + II$$

$$|I| \leq C\Big(4^m R^{\frac{N+2}{p} - 2} + \|f^2\|_p\Big) |Q_m \cap \{w > k_{m+1}\}|^{1 - \frac{1}{p}}$$

$$|II| \leq C\|f_0\|_p^2 |Q_m \cap \{w > k_{m+1}\}|^{\frac{N+4}{N+2} - \frac{2}{p}} \leq C\|f_0\|_p^2 |Q_m \cap \{w > k_{m+1}\}|^{1 - \frac{1}{p}}$$

Therefore

$$R^{N+2} J_{m+1} \leq C\Big(4^m R^{\frac{N+2}{p} - 2} + \|f^2\|_p + \|f_0\|_p^2\Big) |Q_m \cap \{w > k_{m+1}\}|^{\frac{2}{N+2} + 1 - \frac{1}{p}} \qquad (3.11)$$

Also

$$\iint_{Q_m} (w - k_m)^{+2} dxdt \geq (k_{m+1} - k_m)^2 |Q_m \cap \{w > k_{m+1}\}|$$

so that

$$|Q_m \cap \{w > k_{m+1}\}| \leq \frac{C R^{N+2} 4^m}{M^2} J_m$$

Using this in 3.11 we obtain

$$J_{m+1} \leq \frac{C}{R^{N+2}} \Big(4^m R^{\frac{N+2}{p} - 2} + \|f^2\|_p + \|f_0\|_p^2\Big) \Big(\frac{R^{N+2} 4^m}{M^2} J_m\Big)^{\frac{2}{N+2} + 1 - \frac{1}{p}}$$

$$\|\zeta(w-k)^+\|^2_{V_2(Q(R))} \le C \iint_{\{w>k\}\cap\operatorname{supp}\zeta} (|\nabla\zeta|^2 + |\Delta\zeta| + |\zeta_t| + f^2)\,dxdt$$

$$+ C \iint_{Q(R)} |f_0|(w-k)^+\zeta^2\,dxdt \qquad (3.8)$$

Since $\zeta(w-k)^+ \in \overset{\circ}{V}_2(Q(R))$ we may use Lemma 2.1 to get

$$\|\zeta(w-k)^+\|^2_{L^{2\left(\frac{N+2}{N}\right)}(Q(R))} \le C \iint_{\{w>k\}\cap\operatorname{supp}\zeta} (|\nabla\zeta|^2 + |\Delta\zeta| + |\zeta_t| + f^2)\,dxdt$$

$$+ C\Big(\iint_{\{w>k\}\cap\operatorname{supp}\zeta} |f_0|^{\frac{2N+4}{N+4}}\,dxdt\Big)^{\frac{N+4}{N+2}} \qquad (3.9)$$

This inequality will now be iterated on a sequence of shrinking cylinders. Set $k_m = \frac{M}{2}\left(1 - \frac{1}{2^m}\right)$, $R_m = \frac{R}{2}\left(1 + \frac{1}{2^m}\right)$, $Q_m = Q(R_m)$. Choose smooth test functions $\zeta_m(x,t)$ such that $0 \le \zeta_m \le 1$, $\zeta_m \equiv 1$ on Q_{m+1}, $\zeta_m = 0$ near ∂Q_m, and $|\nabla\zeta_m|^2, |\Delta\zeta_m|, |\zeta_{m+1}| \le \frac{C4^m}{R^2}$. Define

$$J_m = \frac{1}{|Q_0|}\iint_{Q_m} (w-k_m)^{+2}\,dxdt\ .$$

We will show that $\{J_m\}$ satisfies the hypotheses of Lemma 2.2. It will follow that if $J_0 = \frac{1}{|Q(R)|}\iint_{Q(R)} (M-u)^2\,dxdt$ is small enough then $\iint_{Q\left(\frac{R}{2}\right)} \left(\frac{M}{2} - u\right)^{+2}dxdt = 0$ i.e. $u \ge \frac{M}{2}$ on $Q\left(\frac{R}{2}\right)$.

We have

$$|Q_0|J_{m+1} \le \Big(\iint_{Q_{m+1}} (w-k_{m+1})^{+2\left(\frac{N+2}{N}\right)}dxdt\Big)^{\frac{N}{N+2}} |Q_{m+1} \cap \{w > k_{m+1}\}|^{\frac{2}{N+2}}$$

$$\le \Big(\iint_{Q_{m+1}} (w-k_{m+1})^{+2\left(\frac{N+2}{N}\right)}dxdt\Big)^{\frac{N}{N+2}} |Q_m \cap \{w > k_{m+1}\}|^{\frac{2}{N+2}} \qquad (3.10)$$

Let $b_1 = \frac{2}{N+2} - \frac{1}{p} > 0$. The condition on p also implies that the net power of R appearing on the right side is nonnegative. Thus, finally

$$J_{m+1} \le C(1 + \|f^2\|_p + \|f_0\|_p^2) \frac{64^m}{M^{2(1+b_1)}} J_m^{1+b_1} \equiv k_1 (k_2)^m J_m^{1+b_1}$$

It follows that $J_m \to 0$ provided

$$J_0 \le k_1^{-\frac{1}{b_1}} k_2^{-\frac{1}{b_1^2}} \equiv \alpha_1 M^{p_0} \qquad p_0 = 2 + \frac{2}{b_1}$$

Letting $\alpha_0 = \frac{\alpha_1}{2C_2}$, we see that if

$$\frac{1}{|Q(R)|} \iint_{Q(R)} (M - u)\,dxdt \le \alpha_0 M^{p_0}$$

then $u \ge \frac{M}{2}$ in $Q\left(\frac{R}{2}\right)$. //

It follows from this result that if the solution is zero at the vertex of some

Let $b_1 = \frac{2}{N+2} - \frac{1}{p} > 0$. The condition on p also implies that the net power of R appearing on the right side is nonnegative. Thus, finally

$$J_{m+1} \le C(1 + \|f^2\|_p + \|f_0\|_p^2) \frac{64^m}{M^{2(1+b_1)}} J_m^{1+b_1} = k_1(k_2)^m J_m^{1+b_1}$$

It follows that $J_m \to 0$ provided

$$J_0 \le k_1^{-\frac{1}{b_1}} k_2^{-\frac{1}{b_1^2}} \equiv \alpha_1 M^{p_0} \qquad p_0 = 2 + \frac{2}{b_1}$$

Letting $\alpha_0 = \frac{\alpha_1}{2C_2}$, we see that if

$$\frac{1}{|Q(R)|} \iint_{Q(R)} (M - u)\,dxdt \le \alpha_0 M^{p_0}$$

then $u \ge \frac{M}{2}$ in $Q\left(\frac{R}{2}\right)$. //

It follows from this result that if the solution is zero at the vertex of some cylinder then it must be bounded away from its positive maximum on some fixed fraction of the cylinder.

<u>Corollary 3.1</u>. Let u be the solution of 1.5 (n,ε) or else $u = u^\varepsilon = \lim_{n\to\infty} {}^n u^\varepsilon$. Let $Q(R) = Q_{x_0,t_0}(R) \subset Q_T$, $u(x_0,t_0) < \frac{M}{2}$, and $0 < \max_{Q(R)} u \le M \le C_2$.

Then there exist constants p_0, β_1, γ_1 depending only on the data and C_2 such that

$$|Q(R) \cap \{u \le M - \gamma_1 M^{p_0}\}| \ge \beta_1 M^{p_0} |Q(R)| \quad . \qquad (3.12)$$

<u>Proof</u>. If u solves 1.5 (n,ε) then u satisfies the conditions of Proposition 3.1, hence it also satisfies the conclusions of the proposition. Since ${}^n u^\varepsilon \to u^\varepsilon$ uniformly the same is true for u^ε. The constants α_0 and p_0 are independent of n and ε.

Choose β_1 and γ_1 so that $2C_2\beta_1 + \gamma_1 < \alpha_0$ and suppose that 3.12 fails. *Then*

$$\iint\limits_{Q(R)} (M - u)dxdt = \iint\limits_{Q(R)\cap\{u\leq M-\gamma_1 M^{p_0}\}} (M - u)dxdt$$

$$+ \iint\limits_{Q(R)\cap\{u>M-\gamma_1 M^{p_0}\}} (M - u)dxdt$$

Thus

$$\iint\limits_{Q(R)} (M - u)dxdt \leq 2C_2|Q(R) \cap \{u \leq M - \gamma_1 M^{p_0}\}| + \gamma_1 M^{p_0}|Q(R)|$$

$$\leq (2C_2\beta_1 + \gamma_1)M^{p_0}|Q(R)| < \alpha_0 M^{p_0}|Q(R)|$$

By Proposition 3.1

$$u(x_0,t_0) \geq \min_{Q\left(\frac{R}{2}\right)} u(x,t) \geq \frac{M}{2}$$

a contradiction.//

SECTION 4.

Due to the special structure of equation 1.1, the solution $u(x,t)$ is related to a subsolution of a certain non-degenerate linear equation. This fact allows us to exploit known results from the linear theory.

We will say that a function $w \in W^{1,1}(Q_T)$ satisfies $w_t - (a_{ij}w_{x_j})_{x_i} \leq G(x,t)$ in Q_T if

$$\iint\limits_{Q_T} (w_t\psi + a_{ij}w_{x_j}\psi_{x_i})dxdt \leq \iint\limits_{Q_T} G\psi dxdt$$

for all $\psi \in \overset{\circ}{W}{}^{1,0}(Q_T)$, $\psi \geq 0$.

Choose β_1 and γ_1 so that $2C_2\beta_1 + \gamma_1 < \alpha_0$ and suppose that 3.12 *fails*. *Then*

$$\iint_{Q(R)} (M - u)dxdt = \iint_{Q(R)\cap\{u\le M-\gamma_1 M^{p_0}\}} (M - u)dxdt$$

$$+ \iint_{Q(R)\cap\{u>M-\gamma_1 M^{p_0}\}} (M - u)dxdt$$

Thus

$$\iint_{Q(R)} (M - u)dxdt \le 2C_2|Q(R) \cap \{u \le M - \gamma_1 M^{p_0}\}| + \gamma_1 M^{p_0}|Q(R)|$$

$$\le (2C_2\beta_1 + \gamma_1)M^{p_0}|Q(R)| < \alpha_0 M^{p_0}|Q(R)|$$

By Proposition 3.1

$$u(x_0,t_0) \ge \min_{Q\left(\frac{R}{2}\right)} u(x,t) \ge \frac{M}{2}$$

a contradiction.//

SECTION 4.

Due to the special structure of equation 1.1, the solution $u(x,t)$ is related to a subsolution of a certain non-degenerate linear equation. This fact allows us to exploit known results from the linear theory.

We will say that a function $w \in W^{1,1}(Q_T)$ satisfies $w_t - (a_{ij}w_{x_j})_{x_i} \le G(x,t)$ in Q_T if

$$\iint_{Q_T} (w_t\psi + a_{ij}w_{x_j}\psi_{x_i})dxdt \le \iint_{Q_T} G\psi dxdt$$

for all $\psi \in \overset{\circ}{W}{}^{1,0}(Q_T)$, $\psi \ge 0$.

Proposition 4.1. Let $v(x,t)$ be the solution of 1.4 (n,ε) and $z(x,t) = [v(x,t) - \gamma]^+$ where $\gamma > 0$. Then there exists $\delta > 0$ depending only on γ and the data such that

$$z_t \leq \nabla\cdot(\alpha(x,t)\nabla z) + |F_\varepsilon| \tag{4.1}$$

for some function $\alpha(x,t)$ satisfying

$$\delta \leq \alpha(x,t) \leq \frac{1}{\delta}. \tag{4.2}$$

Proof. Clearly $z \in W^{1,1}(Q_T)$. Choose a sequence $g_k \in C^\infty(\mathbf{R})$ satisfying $0 \leq g_k'(s) \leq 1$, $g_k''(s) \geq 0$, $g_k(s) = 0$ for $s \leq \frac{\gamma}{2}$, and $g_k(s) \to [s-\gamma]^+$ uniformly on $\mathbf{R}$. Set $z_k(x,t) = g_k(v(x,t))$.

We have

$$(z_k)_t - \nabla\cdot((\phi_n'(v) + \varepsilon)\nabla z_k) \leq |F_\varepsilon| \tag{4.3}$$

pointwise in Q_T. Set

$$\alpha(x,t) = \begin{cases} \phi_n'(v(x,t)) + \varepsilon & v(x,t) > \frac{\gamma}{2} \\ \phi_n'(\frac{\gamma}{2}) + \varepsilon & v(x,t) \leq \frac{\gamma}{2} \end{cases}$$

Clearly 4.2 is satisfied for some $\delta > 0$ depending only on γ and the data. Using the fact that $(z_k)_t = \nabla z_k = 0$ a.e. on $v \leq \frac{\gamma}{2}$, it follows that

$$\iint_{Q_T} ((z_k)_t\psi + \alpha(x,t)\nabla z_k\cdot\nabla\psi)\,dxdt$$

$$= \iint_{\{v>\frac{\gamma}{2}\}} ((z_k)_t\psi + (\phi_n'(v(x,t)) + \varepsilon)\nabla z_k\cdot\nabla\psi)\,dxdt$$

$$= \iint_{Q_T} ((z_k)_t - \nabla\cdot((\phi_n'(v) + \varepsilon)\nabla z_k))\psi\,dxdt$$

$$\leq \iint_{Q_T} |F_\varepsilon|\psi\,dxdt \qquad \text{by 4.3}$$

if $\psi \in \overset{\circ}{W}{}^{1,0}(Q_T)$, $\psi \geq 0$. Letting $k \to \infty$ we get

$$\iint_{Q_T} (z_t\psi + \alpha(x,t)\nabla z\cdot\nabla\psi)\,dxdt \leq \iint_{Q_T} |F_\varepsilon|\psi\,dxdt$$

Thus 4.1 holds.//

Remarks. (i) $\frac{\partial\alpha}{\partial t} \in L^\infty(Q')$ for any $Q' \subset\subset Q_T$ by our construction. This will be used later.

(ii) It is the presence of $|F_\varepsilon|$ instead of F_ε in this differential inequality which causes the complications in the condition on F.

Next we state a modification of a result due to Kruzkov [13], which says that positive supersolutions of linear, non-degenerate, divergence form equations have the property that if they are greater than one on a certain fraction of a cylinder $Q(R)$, then they are bounded below by a positive constant c_0 on some subcylinder $Q(R')$. A result of this nature is the essential step in a proof of Theorem 2.1.

Proposition 4.2. Let $w \in W^{1,1}(Q(R)) \cap C(\overline{Q(R)})$ satisfy

(i) $w_t - (a_{ij}w_{x_j})_{x_i} \geq 0$ in $Q(R)$

$$\delta|\xi|^2 \leq a_{ij}\xi_i\xi_j, \quad \xi \in \mathbf{R}^N, \quad \|a_{ij}\|_{L^\infty} \leq \frac{1}{\delta}, \quad \delta > 0$$

(ii) $w \geq 0$ in $Q(R)$

(iii) There exists $\beta_2 > 0$ such that

$$|Q(R) \cap \{w \geq 1\}| \geq \beta_2|Q(R)|$$

Then there is a constant $c_0 > 0$ depending only on N, δ and β_2 such that

$$w(x,t) \geq c_0 \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right).$$

Remark. This is proved in [13] under the assumptions that $w_t - (a_{ij}w_{x_j})_{x_i} = 0$ and $\beta_2 = \frac{1}{2}$. The extension to supersolutions is immediate, and the case of $\beta_2 \neq \frac{1}{2}$ is also an easy modification of the proof given there.

Corollary 4.1. Let $z \in W^{1,1}(Q(R)) \cap C(\overline{Q(R)})$ satisfy

(i) $z_t \leq (a_{ij}z_{x_j})_{x_i} + G$ in $Q(R)$, $G \in L^1(Q(R))$

$$\delta|\xi|^2 \leq a_{ij}\xi_i\xi_j, \quad \|a_{ij}\|_{L^\infty} \leq \frac{1}{\delta}, \quad \delta > 0.$$

(ii) $z \leq M$ in $Q(R)$, $M > 0$

Remarks. (i) $\frac{\partial \alpha}{\partial t} \in L^{\infty}(Q')$ for any $Q' \subset\subset Q_T$ by our construction. This will be used later.

(ii) It is the presence of $|F_\varepsilon|$ instead of F_ε in this differential inequality which causes the complications in the condition on F.

Next we state a modification of a result due to Kruzkov [13], which says that positive supersolutions of linear, non-degenerate, divergence form equations have the property that if they are greater than one on a certain fraction of a cylinder $Q(R)$, then they are bounded below by a positive constant c_0 on some subcylinder $Q(R')$. A result of this nature is the essential step in a proof of Theorem 2.1.

Proposition 4.2. Let $w \in W^{1,1}(Q(R)) \cap C(\overline{Q(R)})$ satisfy

(i) $w_t - (a_{ij} w_{x_j})_{x_i} \geq 0$ in $Q(R)$

$$\delta|\xi|^2 \leq a_{ij}\xi_i\xi_j, \quad \xi \in \mathbf{R}^N, \quad \|a_{ij}\|_{L^\infty} \leq \frac{1}{\delta}, \quad \delta > 0$$

(ii) $w \geq 0$ in $Q(R)$

(iii) There exists $\beta_2 > 0$ such that

$$|Q(R) \cap \{w \geq 1\}| \geq \beta_2 |Q(R)|$$

Then there is a constant $c_0 > 0$ depending only on N, δ and β_2 such that

$$w(x,t) \geq c_0 \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right).$$

Remark. This is proved in [13] under the assumptions that $w_t - (a_{ij} w_{x_j})_{x_i} = 0$ and $\beta_2 = \frac{1}{2}$. The extension to supersolutions is immediate, and the case of $\beta_2 \neq \frac{1}{2}$ is also an easy modification of the proof given there.

Corollary 4.1. Let $z \in W^{1,1}(Q(R)) \cap C(\overline{Q(R)})$ satisfy

(i) $z_t \leq (a_{ij} z_{x_j})_{x_i} + G$ in $Q(R)$, $G \in L^1(Q(R))$

$$\delta|\xi|^2 \leq a_{ij}\xi_i\xi_j, \quad \|a_{ij}\|_{L^\infty} \leq \frac{1}{\delta}, \quad \delta > 0.$$

(ii) $z \leq M$ in $Q(R)$, $M > 0$

(iii) $|Q(R) \cap \{z < \frac{M}{2}\}| \geq \beta_2 |Q(R)|$ for some $\beta_2 > 0$.

Suppose also

(iv) The equation

$$\hat{z}_t = (a_{ij}\hat{z}_{x_j})_{x_i} + G$$

has a solution $\hat{z} \in W^{1,1}(Q(R)) \cap C(\overline{Q(R)})$.

Then there is a constant $c_0 > 0$ depending on N, δ and β_2 such that

$$z \leq M - Mc_0 + 2\|\hat{z}\|_{L^\infty(Q(R))} \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right) .$$

<u>**Proof**</u>. Apply Proposition 4.2 to the function

$$w = \frac{2}{M}\,(M + \hat{z} + \|\hat{z}\|_{L^\infty} - z) \ . \ //$$

We are now prepared to present the main step in the shrinking cylinder argument. Let $Q_d = \{(x,t) \in Q_T : \mathrm{dist}((x,t),\partial Q_T) \geq d\}$. Fix $d > 0$, $(x_0,t_0) \in Q_d$, and let $Q(R) = Q_{x_0,t_0}(R)$. For the rest of the section we let $\varepsilon \leq \varepsilon_0$ where ε_0 is the constant corresponding to $\frac{d}{2}$ from H2.

<u>Lemma 4.1</u>. Let $v(x,t)$ be the solution of 2.12 (ε) constructed in section 2. Let γ, M and β_2 be given positive numbers. Put $z = [v - \gamma]^+$. Then there are constants R^* and $\tilde{\sigma}$ depending on γ, M, β_2 and the data, $0 < R^* < \frac{d}{2}$, $0 < \tilde{\sigma} < \frac{M}{2}$, such that the following is true for $R < R^*$.

If (i) $z \leq M$ in $Q(R)$

(ii) $|Q(R) \cap \{z = 0\}| \geq \beta_2 |Q(R)|$

then

$$z \leq M - \tilde{\sigma} \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right) .$$

<u>**Proof**</u>. Let $z^n = [v^n - \gamma]^+$ where v^n is the solution of 1.4 (n,ε). Since $z^n \to z$ uniformly on $Q(R)$ there is a sequence $M^n \to M$, $M^n > M$ such that $z^n \leq M^n$ in $Q(R)$. Also for n large enough

$$Q(R) \cap \{z = 0\} \subset Q(R) \cap \{z^n < \frac{M}{2}\} \subset Q(R) \cap \{z^n < \frac{M^n}{2}\} .$$

(iii) $|Q(R) \cap \{z < \frac{M}{2}\}| \geq \beta_2 |Q(R)|$ for some $\beta_2 > 0$.

Suppose also

(iv) The equation

$$\hat{z}_t = (a_{ij}\hat{z}_{x_j})_{x_i} + G$$

has a solution $\hat{z} \in W^{1,1}(Q(R)) \cap C(\overline{Q(R)})$.

Then there is a constant $c_0 > 0$ depending on N, δ and β_2 such that

$$z \leq M - Mc_0 + 2\|\hat{z}\|_{L^\infty(Q(R))} \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right) .$$

Proof. Apply Proposition 4.2 to the function

$$w = \frac{2}{M}\,(M + \hat{z} + \|\hat{z}\|_{L^\infty} - z) \ . \ //$$

We are now prepared to present the main step in the shrinking cylinder argument. Let $Q_d = \{(x,t) \in Q_T : \mathrm{dist}((x,t),\partial Q_T) > d\}$. Fix $d > 0$, $(x_0,t_0) \in Q_d$, and let $Q(R) = Q_{x_0,t_0}(R)$. For the rest of the section we let $\varepsilon < \varepsilon_0$ where ε_0 is the constant corresponding to $\frac{d}{2}$ from H2.

Lemma 4.1. Let $v(x,t)$ be the solution of 2.12 (ε) constructed in section 2. Let γ, M and β_2 be given positive numbers. Put $z = [v - \gamma]^+$. Then there are constants R^* and $\tilde{\sigma}$ depending on γ, M, β_2 and the data, $0 < R^* < \frac{d}{2}$, $0 < \tilde{\sigma} < \frac{M}{2}$, such that the following is true for $R < R^*$.

If (i) $z \leq M$ in $Q(R)$

(ii) $|Q(R) \cap \{z = 0\}| \geq \beta_2 |Q(R)|$

then

$$z \leq M - \tilde{\sigma} \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right) .$$

Proof. Let $z^n = [v^n - \gamma]^+$ where v^n is the solution of 1.4 (n,ε). Since $z^n \to z$ uniformly on $Q(R)$ there is a sequence $M^n \to M$, $M^n > M$ such that $z^n \leq M^n$ in $Q(R)$. Also for n large enough

$$Q(R) \cap \{z = 0\} \subset Q(R) \cap \{z^n < \frac{M}{2}\} \subset Q(R) \cap \{z^n < \frac{M^n}{2}\} .$$

By Proposition 4.1 we have

$$(z^n)_t - \nabla\cdot(\alpha(x,t)\nabla z^n) \le |F_\varepsilon| \le G_\varepsilon$$

where $\delta \le \alpha(x,t) \le \frac{1}{\delta}$ and $\delta > 0$ depends only on γ and the data.

Let $\hat{z}$ be the solution of

$$\hat{z}_t = \nabla\cdot(\alpha(x,t)\nabla\hat{z}) + G_\varepsilon$$

$$\hat{z}|_{\partial Q(R)} = 0$$

The properties of $\alpha(x,t)$ and G_ε guarantee the existence of $\hat{z}$. We noted earlier that $\frac{\partial\alpha}{\partial t} \in L^\infty(Q(R))$, and this is enough to ensure that $\hat{z} \in W^{1,1}(Q(R))$, by Theorem 6.1, Chapter III of [14]. By Theorem 10.1, Chapter III of [14] $\hat{z} \in C(\overline{Q(R)})$. By Corollary 2.2

$$\|\hat{z}\|_{L^\infty(Q(R))} \le \tilde{C}(\text{data},\gamma,R) .$$

Since also $z^n \in W^{1,1}(Q(R)) \cap C(\overline{Q(R)})$ we may apply Corollary 4.1. We conclude

$$z^n \le M^n - M^n c_0 + 2\|\hat{z}\|_{L^\infty(Q(R))} \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right)$$

for some constant c_0 depending only on N, γ and β_2. *We now pick* $R^* > 0$ *such that*

$$\tilde{C}(\text{data},\gamma,R^*) \le \frac{Mc_0}{4} .$$

This choice depends on γ, M, β_2 and the data. Thus, for $R < R^*$ we have

$$z^n \le M^n - M^n c_0 + \frac{Mc_0}{2} \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right) .$$

Letting $n \to \infty$

$$z \le M - \frac{Mc_0}{2} \equiv M - \tilde{\sigma} \quad \text{in} \quad Q\left(\frac{\beta_2 R}{8}\right) . \quad //$$

<u>Remark</u>. This proof could be simplified if it were known that Proposition 4.2 is true assuming only $w \in W^{1,0}(Q(R))$. This is done in [15]. It is then possible to work directly with z instead of the approximations z^n.

<u>Proposition 4.3</u>. Let v be the solution of 2.12 (ε) and $u = \phi(v)$. Suppose $u(x_0,t_0) = 0$, $(x_0,t_0) \in Q_d$. Then there exist sequences $M_k \downarrow 0$, $R_k \downarrow 0$ depending only

on d and the data, such that

$$\sup_{Q(R_j)} |u(x,t)| \leq M_j$$

Proof. Define

$$\gamma(M) = \beta(M - \gamma_1 M^{p_0})$$

$$\sigma(M) = \tilde{\sigma}(\gamma(M), \beta(M) - \gamma(M), \beta_1 M^{p_0})$$

$$R^*(M) = R^*(\gamma(M), \beta(M) - \gamma(M), \beta_1 M^{p_0})$$

where γ_1, β_1 and p_0 are the constants from Corollary 3.1 and $\tilde{\sigma}(\gamma, M, \beta_2)$, $R^*(\gamma, M, \beta_2)$ are the constants from Lemma 4.2. Set

$$M_1 = \|u\|_{L^\infty(Q_T)}$$

$$M_{k+1} = \phi(\beta(M_k) - \sigma(M_k))$$

$$R_1 = R^*(M_1)$$

$$R_{k+1} = \min\left(R^*(M_{k+1}), \frac{\beta_1 M_k^{p_0} R_k}{8}\right)$$

Clearly $M_k \to 0$, $R_k \to 0$. We now show by induction that $|u| \leq M_j$ on $Q(R_j)$. This is clear for $j = 1$. Assume, then, that

$$\sup_{Q(R_k)} |u| \leq M_k \tag{4.4}$$

Set $z = [v - \gamma(M_k)]^+$, so that

$$z \leq \beta(M_k) - \gamma(M_k)$$

By Corollary 3.1

$$|Q(R_k) \cap \{z = 0\}| = |Q(R_k) \cap \{u \leq M_k - \gamma_1 M_k^{p_0}\}| \geq \beta_1 M_k^{p_0} |Q(R_k)| .$$

We may now apply Lemma 4.1 with γ, M, β_2 replaced by $\gamma(M_k)$, $\beta(M_k) - \gamma(M_k)$ and $\beta_1 M_k^{p_0}$. Note that $R_k \leq R^*\left(\gamma(M_k), \beta(M_k) - \gamma(M_k), \beta_1 M_k^{p_0}\right)$. We conclude that

on d and the data, such that

$$\sup_{Q(R_j)} |u(x,t)| \leq M_j$$

Proof. Define

$$\gamma(M) = \beta(M - \gamma_1 M^{p_0})$$

$$\sigma(M) = \tilde{\sigma}(\gamma(M), \beta(M) - \gamma(M), \beta_1 M^{p_0})$$

$$R^*(M) = R^*(\gamma(M), \beta(M) - \gamma(M), \beta_1 M^{p_0})$$

where γ_1, β_1 and p_0 are the constants from Corollary 3.1 and $\tilde{\sigma}(\gamma, M, \beta_2)$, $R^*(\gamma, M, \beta_2)$ are the constants from Lemma 4.2. Set

$$M_1 = \|u\|_{L^\infty(Q_T)}$$

$$M_{k+1} = \phi(\beta(M_k) - \sigma(M_k))$$

$$R_1 = R^*(M_1)$$

$$R_{k+1} = \min\left(R^*(M_{k+1}), \frac{\beta_1 M_k^{p_0} R_k}{8}\right)$$

Clearly $M_k \to 0$, $R_k \to 0$. We now show by induction that $|u| \leq M_j$ on $Q(R_j)$. This is clear for $j = 1$. Assume, then, that

$$\sup_{Q(R_k)} |u| \leq M_k \tag{4.4}$$

Set $z = [v - \gamma(M_k)]^+$, so that

$$z \leq \beta(M_k) - \gamma(M_k)$$

By Corollary 3.1

$$|Q(R_k) \cap \{z = 0\}| = |Q(R_k) \cap \{u \leq M_k - \gamma_1 M_k^{p_0}\}| \geq \beta_1 M_k^{p_0} |Q(R_k)| .$$

We may now apply Lemma 4.1 with γ, M, β_2 replaced by $\gamma(M_k)$, $\beta(M_k) - \gamma(M_k)$ and $\beta_1 M_k^{p_0}$. Note that $R_k \leq R^*\left(\gamma(M_k), \beta(M_k) - \gamma(M_k), \beta_1 M_k^{p_0}\right)$. We conclude that

$$z \leq \beta(M_k) - \gamma(M_k) - \sigma(M_k) \quad \text{in} \quad Q\left(\frac{\beta_1 M_k^{p_0} R_k}{8}\right) . \tag{4.5}$$

Therefore

$$v \leq \beta(M_k) - \sigma(M_k) \quad \text{in} \quad Q(R_{k+1})$$

and so

$$u \leq \phi(\beta(M_k) - \sigma(M_k)) = M_{k+1} \quad \text{in} \quad Q(R_{k+1}) .$$

The same argument applies to $-u$, hence 4.4 holds with k replaced by $k + 1$.//

The content of the last proposition is essentially a modulus of continuity from below at any point where the solution vanishes. We wish now to examine the behavior of the solution in the vicinity of a point where it is not zero. The next result states that in some full neighborhood of such a point the solution must be bounded away from zero.

Define

$$Q'_{x_0,t_0}(R) = \{(x,t) : |x - x_0| < R, t_0 - R^2 < t < t_0 + R^2\}$$

<u>Proposition 4.4</u>. Let u and v be as in Proposition 4.3 and let $\{M_k\}$ and $\{R_k\}$ be the sequences given there. Let $(x_0,t_0) \in Q_{2d}$ and suppose

$$M_{k_0+1} < u(x_0,t_0) \leq M_{k_0}$$

for some k_0. Put

$$\bar{R}_{k_0} = \min\left(R_{k_0+1}, \frac{R_{k_0}}{2}\right)$$

Then

$$u \geq \frac{M_{k_0}}{2} \quad \text{in} \quad Q'_{x_0,t_0}(\bar{R}_{k_0})$$

<u>Proof</u>. The proof of Proposition 4.3 would show that $u(x_0,t_0) \leq \sup_{Q(R_{k_0+1})} u \leq M_{k_0+1}$ unless the induction hypotheses fail by the k_0'th step. The only way this can happen is if

$$|Q(R_k) \cap \{u \leq M_k - \gamma_1 M_k^{p_0}\}| < \beta_1 M_k^{p_0} |Q(R_k)|$$

$$z \leq \beta(M_k) - \gamma(M_k) - \sigma(M_k) \quad \text{in} \quad Q\left(\frac{\beta_1 M_k^{p_0} R_k}{8}\right) . \tag{4.5}$$

Therefore

$$v \leq \beta(M_k) - \sigma(M_k) \quad \text{in} \quad Q(R_{k+1})$$

and so

$$u \leq \phi(\beta(M_k) - \sigma(M_k)) = M_{k+1} \quad \text{in} \quad Q(R_{k+1}) .$$

The same argument applies to $-u$, hence 4.4 holds with k replaced by $k + 1$.//

The content of the last proposition is essentially a modulus of continuity from below at any point where the solution vanishes. We wish now to examine the behavior of the solution in the vicinity of a point where it is not zero. The next result states that in some full neighborhood of such a point the solution must be bounded away from zero.

Define

$$Q'_{x_0,t_0}(R) = \{(x,t) : |x - x_0| < R, t_0 - R^2 < t < t_0 + R^2\}$$

<u>Proposition 4.4</u>. Let u and v be as in Proposition 4.3 and let $\{M_k\}$ and $\{R_k\}$ be the sequences given there. Let $(x_0,t_0) \in Q_{2d}$ and suppose

$$M_{k_0+1} < u(x_0,t_0) \leq M_{k_0}$$

for some k_0. Put

$$\bar{R}_{k_0} = \min\left(R_{k_0+1}, \frac{R_{k_0}}{2}\right)$$

Then
$$u > \frac{M_{k_0}}{2} \quad \text{in} \quad Q'_{x_0,t_0}\left(\bar{R}_{k_0}\right)$$

<u>Proof</u>. The proof of Proposition 4.3 would show that $u(x_0,t_0) \leq \sup\limits_{Q(R_{k_0+1})} u \leq M_{k_0+1}$ unless the induction hypotheses fail by the k_0'th step. The only way this can happen is if

$$|Q(R_k) \cap \{u \leq M_k - \gamma_1 M_k^{p_0}\}| < \beta_1 M_k^{p_0} |Q(R_k)|$$

for some $k \leq k_0$. As in the proof of Corollary 3.1 this implies that $u \geq \frac{M_k}{2}$ in $Q(\frac{R_k}{2})$.

Therefore $u \geq \frac{M_{k_0}}{2}$ in $Q(\bar{R}_{k_0})$ (i.e. in the backward cylinder with vertex (x_0,t_0)).

Now suppose that $u(x_1,t_1) < \frac{M_{k_0}}{2}$ for some $(x_1,t_1) \in Q'(\bar{R}_{k_0})$ with $t_1 > t_0$. Note that $(x_0,t_0) \in Q_{x_1,t_1}(R_{k_0+1})$ and $(x_1,t_1) \in Q_d$. Again the induction argument of Proposition 4.3 will work up to the k_0'th step, showing that

$$u \leq M_{k_0+1} \quad \text{in} \quad Q_{x_1,t_1}(R_{k_0+1})$$

a contradiction.//

Proposition 4.5. The functions $u^\varepsilon(x,t)$ are equicontinuous on Q_{2d} for any $d > 0$, $\varepsilon \leq \varepsilon_0$.

Proof. Fix $d > 0$, $\varepsilon \leq \varepsilon_0$ and $\eta > 0$. We must find $\delta > 0$ depending only on η, d and the data such that

$$|u^\varepsilon(x_1,t_1) - u^\varepsilon(x_0,t_0)| \leq \eta \tag{4.6}$$

whenever $(x_1,t_1),(x_0,t_0) \in Q_{2d}$ and

$$|x_1 - x_0|^2 + |t_1 - t_0| < \delta .$$

Let $L = \max(|u^\varepsilon(x_1,t_1)|,|u^\varepsilon(x_0,t_0)|)$. If $L \leq \frac{\eta}{2}$ there is nothing to prove, so assume that

$$L = u^\varepsilon(x_0,t_0) > \frac{\eta}{2} .$$

(Otherwise apply the same argument to $-u^\varepsilon$). By Proposition 4.4 $u^\varepsilon(x,t) \geq \frac{\eta}{4}$ on $Q'_{x_0,t_0}(\bar{R})$ where $\bar{R}$ depends on η, d and the data. On this cylinder $v^\varepsilon(x,t)$ satisfies the linear equation

$$v_t = \nabla\cdot(\alpha(x,t)\nabla v) + f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$$

where $\alpha(x,t) = \phi'(v(x,t)) + \varepsilon \geq \phi'(\beta(\frac{\eta}{4})) + \varepsilon \geq \theta > 0$ where θ is a constant depending only on η and the data.

By Theorem 2.1 v^ε is Holder continuous on $Q'_{x_0,t_0}(\frac{\bar{R}}{2})$ with a modulus of continuity depending only on d,η and the data. The quantity d in the statement of Theorem 2.1 is here replaced by $\frac{\bar{R}}{2}$. Since ϕ is locally Lipschitz the same is true for u^ε. Thus we

for some $k < k_0$. As in the proof of Corollary 3.1 this implies that $u > \frac{M_k}{2}$ in $Q(\frac{R_k}{2})$.

Therefore $u > \frac{M_{k_0}}{2}$ in $Q(\bar{R}_{k_0})$ (i.e. in the backward cylinder with vertex (x_0,t_0)).

Now suppose that $u(x_1,t_1) < \frac{M_{k_0}}{2}$ for some $(x_1,t_1) \in Q'(\bar{R}_{k_0})$ with $t_1 > t_0$. Note that $(x_0,t_0) \in Q_{x_1,t_1}(R_{k_0+1})$ and $(x_1,t_1) \in Q_d$. Again the induction argument of Proposition 4.3 will work up to the k_0'th step, showing that

$$u < M_{k_0+1} \quad \text{in} \quad Q_{x_1,t_1}(R_{k_0+1})$$

a contradiction.//

<u>Proposition 4.5</u>. The functions $u^{\varepsilon}(x,t)$ are equicontinuous on Q_{2d} for any $d > 0$, $\varepsilon < \varepsilon_0$.

<u>Proof</u>. Fix $d > 0$, $\varepsilon < \varepsilon_0$ and $\eta > 0$. We must find $\delta > 0$ depending only on η, d and the data such that

$$|u^{\varepsilon}(x_1,t_1) - u^{\varepsilon}(x_0,t_0)| < \eta \tag{4.6}$$

whenever $(x_1,t_1),(x_0,t_0) \in Q_{2d}$ and

$$|x_1 - x_0|^2 + |t_1 - t_0| < \delta .$$

Let $L = \max(|u^{\varepsilon}(x_1,t_1)|,|u^{\varepsilon}(x_0,t_0)|)$. If $L < \frac{\eta}{2}$ there is nothing to prove, so assume that

$$L = u^{\varepsilon}(x_0,t_0) > \frac{\eta}{2} .$$

(Otherwise apply the same argument to $-u^{\varepsilon}$). By Proposition 4.4 $u^{\varepsilon}(x,t) > \frac{\eta}{4}$ on $Q'_{x_0,t_0}(\bar{R})$ where $\bar{R}$ depends on η, d and the data. On this cylinder $v^{\varepsilon}(x,t)$ satisfies the linear equation

$$v_t = \nabla\cdot(\alpha(x,t)\nabla v) + f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$$

where $\alpha(x,t) = \phi'(v(x,t)) + \varepsilon > \phi'(\beta(\frac{\eta}{4})) + \varepsilon > \theta > 0$ where θ is a constant depending only on η and the data.

By Theorem 2.1 v^{ε} is Holder continuous on $Q'_{x_0,t_0}(\frac{\bar{R}}{2})$ with a modulus of continuity depending only on d,η and the data. The quantity d in the statement of Theorem 2.1 is here replaced by $\frac{\bar{R}}{2}$. Since ϕ is locally Lipschitz the same is true for u^{ε}. Thus we

choose δ so that $\delta < \frac{\bar{R}}{2}$ and then again smaller so that 4.6 is satisfied, independently of ε. //

We may now finish the proof of Theorem 1.1. Since β is uniformly continuous on $[-C_2, C_2]$, the functions u^ε and v^ε, $\varepsilon < \varepsilon_0$ are uniformly bounded and equicontinuous on Q_d for any $d > 0$. Thus we may find limit function

$$u(x,t) = \lim_{\varepsilon_k \to 0} u^{\varepsilon_k}(x,t)$$

$$v(x,t) = \lim_{\varepsilon_k \to 0} v^{\varepsilon_k}(x,t)$$

where the convergence is uniform on compact subsets of Q_T. Clearly $v(x,t) = \beta(u(x,t))$ and $v(x,t)$ and $u(x,t)$ are continuous on Q_T. Also, by Proposition 2.2

$$\nabla u^{\varepsilon_k} \to \nabla u \quad \text{weakly in} \quad L^2(Q_T)$$

for some further subsequence.

If $\psi \in E \cap C^2(Q_T)$ then equation 2.11 for $\varepsilon = \varepsilon_k$ is

$$\iint_{Q_T} (v^{\varepsilon_k}\psi_t - \nabla u^{\varepsilon_k} \cdot \nabla\psi + \varepsilon_k v^{\varepsilon_k}\Delta\psi + f_{0\varepsilon_k}\psi - f_{i\varepsilon_k}\psi_{x_i})dxdt + \int_\Omega v_{0\varepsilon_k}\psi dx = 0$$

Letting $k \to \infty$ gives

$$\iint_{Q_T} (v\psi_t - \nabla u \cdot \nabla\psi + f_0\psi - f_i\psi_{x_i})dxdt + \int_\Omega v_0(x)\psi(x,0)dx = 0$$

This identity remains true for all $\psi \in E$. Hence u is a solution of 1.1 and $u \in L^\infty(Q_T) \cap C(Q_T)$.

<u>Remark</u>. The modulus of continuity depends on the sequence $\{M_k\}$ from Proposition 4.3, which is generated recursively according to the rule

$$M_{k+1} = \phi(\beta(M_k) - \sigma(M_k))$$

with $M_1 = \|u\|_{L^\infty} \le C_2$. This clearly depends on β and $\phi = \beta^{-1}$ (along with all the other data), but it remains to be seen that it may be taken to depend only on the functions

choose δ so that $\delta < \frac{\bar{R}}{2}$ and then again smaller so that 4.6 is satisfied, independently of ε. //

We may now finish the proof of Theorem 1.1. Since β is uniformly continuous on $[-C_2,C_2]$, the functions u^{ε} and v^{ε}, $\varepsilon < \varepsilon_0$ are uniformly bounded and equicontinuous on Q_d for any $d > 0$. Thus we may find limit function

$$u(x,t) = \lim_{\varepsilon_k \to 0} u^{\varepsilon_k}(x,t)$$

$$v(x,t) = \lim_{\varepsilon_k \to 0} v^{\varepsilon_k}(x,t)$$

where the convergence is uniform on compact subsets of Q_T. Clearly $v(x,t) = \beta(u(x,t))$ and $v(x,t)$ and $u(x,t)$ are continuous on Q_T. Also, by Proposition 2.2

$$\nabla u^{\varepsilon_k} \to \nabla u \quad \text{weakly in} \quad L^2(Q_T)$$

for some further subsequence.

If $\psi \in E \cap C^2(Q_T)$ then equation 2.11 for $\varepsilon = \varepsilon_k$ is

$$\iint_{Q_T} (v^{\varepsilon_k}\psi_t - \nabla u^{\varepsilon_k}\cdot\nabla\psi + \varepsilon_k v^{\varepsilon_k}\Delta\psi + f_{0\varepsilon_k}\psi - f_{i\varepsilon_k}\psi_{x_i})dxdt + \int_{\Omega} v_{0\varepsilon_k}\psi dx = 0$$

Letting $k \to \infty$ gives

$$\iint_{Q_T} (v\psi_t - \nabla u\cdot\nabla\psi + f_0\psi - f_i\psi_{x_i})dxdt + \int_{\Omega} v_0(x)\psi(x,0)dx = 0$$

This identity remains true for all $\psi \in E$. Hence u is a solution of 1.1 and $u \in L^{\infty}(Q_T) \cap C(Q_T)$.

<u>Remark</u>. The modulus of continuity depends on the sequence $\{M_k\}$ from Proposition 4.3, which is generated recursively according to the rule

$$M_{k+1} = \phi(\beta(M_k) - \sigma(M_k))$$

with $M_1 = \|u\|_{L^{\infty}} \le C_2$. This clearly depends on β and $\phi = \beta^{-1}$ (along with all the other data), but it remains to be seen that it may be taken to depend only on the functions

μ_1 and μ_2 from H3. Specifically, we must show that there are sequences $M_k' \downarrow 0$, $M_k'' \downarrow 0$ depending only on μ_1, μ_2 and the rest of the data, such that

$$M_k'' \leq M_k \leq M_k' \tag{4.7}$$

To do this we write

$$M_{k+1} = M_k - \rho(M_k)$$

where $\rho(M) = M - \phi(\beta(M) - \sigma(M)) = \phi(\beta(M)) - \phi(\beta(M) - \sigma(M))$. We may assume that $M - \sigma(M)$ is nondecreasing on $[0,C_2]$; if not, replace $\sigma(M)$ by $\bar{\sigma}(M) = \frac{1}{M}\int_0^M \min(\sigma(r), r\beta'(r))dr$. Then $\bar{\sigma}$ satisfies $0 < \bar{\sigma}(M) \leq \sigma(M)$, $\bar{\sigma}'(M) \leq \beta'(M)$. Therefore $\beta(M) - \bar{\sigma}(M)$ is nondecreasing.

Next

$$\rho(M) \geq \Big(\inf_{s \geq \frac{\beta(M)}{2}} \phi'(s)\Big)\sigma(M) \geq \Big(\inf_{s \geq \frac{M\mu_1(M)}{2}} \phi'(s)\Big)\sigma(M) \geq \mu_2\Big(\frac{M\mu_1(M)}{2}\Big)\sigma(M) \equiv \rho_1(M)$$

and

$$\rho(M) \leq \Big(\sup_{s \leq \beta(M)} \phi'(s)\Big)\sigma(M) = \Big(\sup_{s \leq \beta(M)} \frac{1}{\beta'(\phi(s))}\Big)\sigma(M) \leq \frac{1}{\mu_1(C_2)}\,\sigma(M) \equiv \rho_2(M)$$

Then ρ_1 and ρ_2 are nondecreasing functions of M with $\rho_1(0) = \rho_2(0) = 0$. Generate sequences M_k', M_k'' by

$$M_1' = M_1'' = M_1$$

$$M_{k+1}' = M_k' - \rho_1(M_k')$$

$$M_{k+1}'' = M_k'' - \rho_2(M_k'')$$

Then 4.7 clearly holds for $k = 1$; if we assume its validity for some k, then

$$M_{k+1} = M_k - \rho(M_k) \leq M_k' - \rho(M_k') \leq M_k' - \rho_1(M_k') = M_{k+1}'$$

$$M_{k+1} = M_k - \rho(M_k) \geq M_k'' - \rho(M_k'') \geq M_k'' - \rho_2(M_k'') = M_{k+1}''$$

Thus 4.7 holds for all k which was to be shown.

μ_1 and μ_2 from H3. Specifically, we must show that there are sequences $M_k' \downarrow 0$, $M_k'' \downarrow 0$ depending only on μ_1, μ_2 and the rest of the data, such that

$$M_k'' \leq M_k \leq M_k' \tag{4.7}$$

To do this we write

$$M_{k+1} = M_k - \rho(M_k)$$

where $\rho(M) = M - \phi(\beta(M) - \sigma(M)) = \phi(\beta(M)) - \phi(\beta(M) - \sigma(M))$. We may assume that $M - \sigma(M)$ is nondecreasing on $[0,C_2]$; if not, replace $\sigma(M)$ by $\bar{\sigma}(M) = \frac{1}{M}\int_0^M \min(\sigma(r), r\beta'(r))dr$. Then $\bar{\sigma}$ satisfies $0 < \bar{\sigma}(M) \leq \sigma(M)$, $\bar{\sigma}'(M) \leq \beta'(M)$. Therefore $\beta(M) - \bar{\sigma}(M)$ is nondecreasing.

Next

$$\rho(M) \geq \Big(\inf_{s \geq \frac{\beta(M)}{2}} \phi'(s)\Big)\sigma(M) \geq \Big(\inf_{s \geq \frac{M\mu_1(M)}{2}} \phi'(s)\Big)\sigma(M) \geq \mu_2\Big(\frac{M\mu_1(M)}{2}\Big)\sigma(M) \equiv \rho_1(M)$$

and

$$\rho(M) \leq \Big(\sup_{s \leq \beta(M)} \phi'(s)\Big)\sigma(M) = \Big(\sup_{s \leq \beta(M)} \frac{1}{\beta'(\phi(s))}\Big)\sigma(M) \leq \frac{1}{\mu_1(C_2)}\,\sigma(M) \equiv \rho_2(M)$$

Then ρ_1 and ρ_2 are nondecreasing functions of M with $\rho_1(0) = \rho_2(0) = 0$. Generate sequences M_k', M_k'' by

$$M_1' = M_1'' = M_1$$

$$M_{k+1}' = M_k' - \rho_1(M_k')$$

$$M_{k+1}'' = M_k'' - \rho_2(M_k'')$$

Then 4.7 clearly holds for $k = 1$; if we assume its validity for some k, then

$$M_{k+1} = M_k - \rho(M_k) \leq M_k' - \rho(M_k') \leq M_k' - \rho_1(M_k') = M_{k+1}'$$

$$M_{k+1} = M_k - \rho(M_k) \geq M_k'' - \rho(M_k'') \geq M_k'' - \rho_2(M_k'') = M_{k+1}''$$

Thus 4.7 holds for all k which was to be shown.

SECTION 5.

We turn now to the Cauchy problem

$$[\beta(u)]_t = \Delta u + F \qquad (x,t) \in R_T = \mathbf{R}^N \times (0,T) \tag{5.1}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbf{R}^N$$

By a solution of 5.1 we mean a function $u \in V_{2,loc}(R_T)$ such that there exists $v \in L^1_{loc}(R_T)$, $v(x,t) = \beta(u(x,t))$ a.e. satisfying

$$\iint_{R_T} (v\psi_t - \nabla u\cdot\nabla\psi + f_0\psi - f_i\psi_{x_i})dxdt + \int_{\mathbf{R}^N} v_0(x)\psi(x,0)dx = 0 \tag{5.2}$$

for every $\psi \in C_0^\infty(\mathbb{R}^N \times [0,T))$. Here $v_0(x) = \beta(u_0(x))$ and F has the form $f_0 + (f_i)_{x_i}$.

We retain the assumptions H1 and H3 on $u_0(x)$ and β. In place of H2 we assume

H4 (i) $F \in \hat{G}_p(R_T)$ $\quad p > \frac{N+2}{2}$

(ii) If $F_\varepsilon = f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$ is the function from the definition of $\hat{G}_p$, then $\|f_{i\varepsilon}\|_{L^p(R_T)} \le A$ for some constant A independent of ε. Also $p > 2$.

Remark. The proof of Theorem 1.1 is essentially local in nature once we have the global bounds obtained in Section 2. Thus we must show how these results can be modified for the case of an unbounded domain so that the remainder of the proof may proceed as before. By $\hat{G}_p(R_T)$ we mean $\{F \in G_p(R_T)$: there exists F_ε, $G_\varepsilon \in C^1(R_T)$ and $A < \infty$ satisfying

(i) $F_\varepsilon \to F$ in $G_p(R_T)$ as $\varepsilon \to 0$

(ii) $\|F_\varepsilon\|_{G_p}$, $\|G_\varepsilon\|_{G_p} \le A$

(iii) For any $Q' \subset\subset R_T$ there exists $\varepsilon_0 > 0$ such that $|F_\varepsilon| \le G_\varepsilon$ in Q' for $\varepsilon \le \varepsilon_0$.$\}$ Condition (ii) in H4 is used only to obtain a global L^∞ bound; the assumption that $p > 2$ is a restriction only for $N = 1$. The condition H4 is satisfied by any $F \in L^p(R_T)$ for $p > \frac{N+2}{2}$.

Theorem 5.1. Under assumptions H1, H3 and H4 the problem 5.1 has a solution

$$u \in L^\infty(R_T) \cap C(R_T)$$

The norm $\|u\|_{L^\infty(R_T)}$ and the modulus of continuity of u depend only on N, T and the constants from H1, H3 and H4.

SECTION 5.

We turn now to the Cauchy problem

$$[\beta(u)]_t = \Delta u + F \qquad (x,t) \in R_T = \mathbb{R}^N \times (0,T) \tag{5.1}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}^N$$

By a solution of 5.1 we mean a function $u \in V_{2,loc}(R_T)$ such that there exists $v \in L^1_{loc}(R_T)$, $v(x,t) = \beta(u(x,t))$ a.e. satisfying

$$\iint_{R_T} (v\psi_t - \nabla u \cdot \nabla\psi + f_0\psi - f_i\psi_{x_i})\,dxdt + \int_{\mathbb{R}^N} v_0(x)\psi(x,0)\,dx = 0 \tag{5.2}$$

for every $\psi \in C_0^\infty(\mathbb{R}^N \times [0,T))$. Here $v_0(x) = \beta(u_0(x))$ and F has the form $f_0 + (f_i)_{x_i}$.

We retain the assumptions H1 and H3 on $u_0(x)$ and β. In place of H2 we assume

H4 (i) $F \in \hat{G}_p(R_T)$ $\quad p > \frac{N+2}{2}$

(ii) If $F_\varepsilon = f_{0\varepsilon} + (f_{i\varepsilon})_{x_i}$ is the function from the definition of $\hat{G}_p$, then $\|f_{i\varepsilon}\|_{L^p(R_T)} \le A$ for some constant A independent of ε. Also $p > 2$.

Remark. The proof of Theorem 1.1 is essentially local in nature once we have the global bounds obtained in Section 2. Thus we must show how these results can be modified for the case of an unbounded domain so that the remainder of the proof may proceed as before. By $\hat{G}_p(R_T)$ we mean $\{F \in G_p(R_T)$: there exists F_ε, $G_\varepsilon \in C^1(R_T)$ and $A < \infty$ satisfying

(i) $F_\varepsilon \to F$ in $G_p(R_T)$ as $\varepsilon \to 0$

(ii) $\|F_\varepsilon\|_{G_p}$, $\|G_\varepsilon\|_{G_p} \le A$

(iii) For any $Q' \subset\subset R_T$ there exists $\varepsilon_0 > 0$ such that $|F_\varepsilon| \le G_\varepsilon$ in Q' for $\varepsilon \le \varepsilon_0$.$\}$ Condition (ii) in H4 is used only to obtain a global L^∞ bound; the assumption that $p > 2$ is a restriction only for $N = 1$. The condition H4 is satisfied by any $F \in L^p(R_T)$ for $p > \frac{N+2}{2}$.

Theorem 5.1. Under assumptions H1, H3 and H4 the problem 5.1 has a solution

$$u \in L^\infty(R_T) \cap C(R_T)$$

The norm $\|u\|_{L^\infty(R_T)}$ and the modulus of continuity of u depend only on N, T and the constants from H1, H3 and H4.

Remark. (i) Existence results are known from nonlinear semigroup theory for the case $F \in L^1(R_T)$, $(u_0(x)) \in L^1(\mathbf{R}^N)$. One of the earliest existence results for equations of this kind is given in Sabinina [16], for the case $F = 0$, $u_0(x) \in L^2(\mathbf{R}^N)$ with $u_0(x) \geq 0$. See also [17].

(ii) If it is assumed that $\beta(u_0) \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ then uniqueness follows from [8]. The uniqueness problem is currently under active investigation.

For the proof we study, as before, the regularized problem

$$v_t = \Delta(\phi_n(v) + \varepsilon v) + F_\varepsilon(x,t) \qquad (x,t) \in R_T \qquad (5.3)\ (n,\varepsilon)$$
$$v(x,0) = v_{0\varepsilon}(x) \qquad x \in \mathbf{R}^N$$

and the corresponding problem for $u = \phi_n(v)$

$$[\beta_n(u)]_t = \Delta(u + \varepsilon\beta_n(u)) + F_\varepsilon(x,t) \qquad (5.4)\ (n,\varepsilon)$$
$$u(x,0) = \phi_n(v_{0\varepsilon}(v))$$

We choose ϕ_n and $v_{0\varepsilon}$ as in Section 2; in this case $v_{0\varepsilon} \to v_0$ in $L^2(\Omega)$ for any bounded $\Omega \subset \mathbf{R}^N$. $F_\varepsilon(x,t)$ is the function from H4.

Let $B_r = \{x : |x| < r\}$ and $K_r = B_r \times (0,T)$. Suppose $v_{0\varepsilon}(x) = 0$ for $|x| \geq r_0$. A solution of 5.3 (n,ε) may be obtained as the pointwise limit of some subequence of $\{v_r\}_{r \geq r_0}$ where v_r satisfies

$$v_t = \Delta(\phi_n(v) + \varepsilon v) + F_\varepsilon(x,t) \qquad (x,t) \in K_r \qquad (5.5)\ (n,\varepsilon,r)$$
$$v(x,0) = v_{0\varepsilon}(x) \qquad x \in B_r$$
$$v(x,t) = 0 \qquad |x| = r$$

For a discussion of this, see pages 492-496 of [14].

We now show that solutions of this problem are uniformly bounded, independently of n, ε and r. The maximum principle of Section 2 relied on the inequality

$$\mu(k_0) < |Q_T|$$

which is obviously of no use here.

Remark. (i) Existence results are known from nonlinear semigroup theory for the case $F \in L^1(R_T)$, $\beta(u_0(x)) \in L^1(\mathbf{R}^N)$. One of the earliest existence results for equations of this kind is given in Sabinina [16], for the case $F = 0$, $u_0(x) \in L^2(\mathbf{R}^N)$ with $u_0(x) \geq 0$. See also [17].

(ii) If it is assumed that $\beta(u_0) \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ then uniqueness follows from [8]. The uniqueness problem is currently under active investigation.

For the proof we study, as before, the regularized problem

$$v_t = \Delta(\phi_n(v) + \varepsilon v) + F_\varepsilon(x,t) \qquad (x,t) \in R_T \qquad (5.3)\ (n,\varepsilon)$$

$$v(x,0) = v_{0\varepsilon}(x) \qquad x \in \mathbf{R}^N$$

and the corresponding problem for $u = \phi_n(v)$

$$[\beta_n(u)]_t = \Delta(u + \varepsilon\beta_n(u)) + F_\varepsilon(x,t) \qquad (5.4)\ (n,\varepsilon)$$

$$u(x,0) = \phi_n(v_{0\varepsilon}(v))$$

We choose ϕ_n and $v_{0\varepsilon}$ as in Section 2; in this case $v_{0\varepsilon} \to v_0$ in $L^2(\Omega)$ for any bounded $\Omega \subset \mathbf{R}^N$. $F_\varepsilon(x,t)$ is the function from H4.

Let $B_r = \{x : |x| < r\}$ and $K_r = B_r \times (0,T)$. Suppose $v_{0\varepsilon}(x) = 0$ for $|x| > r_0$. A solution of 5.3 (n,ε) may be obtained as the pointwise limit of some subequence of $\{v_r\}_{r>r_0}$ where v_r satisfies

$$v_t = \Delta(\phi_n(v) + \varepsilon v) + F_\varepsilon(x,t) \qquad (x,t) \in K_r \qquad (5.5)\ (n,\varepsilon,r)$$

$$v(x,0) = v_{0\varepsilon}(x) \qquad x \in B_r$$

$$v(x,t) = 0 \qquad |x| = r$$

For a discussion of this, see pages 492-496 of [14].

We now show that solutions of this problem are uniformly bounded, independently of n, ε and r. The maximum principle of Section 2 relied on the inequality

$$\mu(k_0) < |Q_T|$$

which is obviously of no use here.

We will use instead the estimate

$$|\{|f| > k\}| \le \left(\frac{\|f\|_p}{k}\right)^p \tag{5.6}$$

which is valid for $f \in L^p(R_T)$.

<u>Proposition 5.1</u>. There exists a constant C_1 depending only on the data, such that

$$|v(x,t)| \le C_1$$

if v is any solution of (5.5) (n,ε,r), $r > r_0$.

<u>Proof</u>. Apply Proposition 2.1 with $Q_T = K_r$ and $k_0 = \max(1, 2\|v_0\|_{L^\infty})$. We obtain the estimate

$$v(x,t) \le C(\text{data}, \mu(k_0))$$

Set $k_1 = \frac{k_0}{2}$. We claim that

$$\|(v - k_1)^+\|_{L^p(K_r)} \le C \tag{5.7}$$

where C depends only on the data. If this is done, then it follows from 5.6 that

$$\mu(k_0) = |\{v > k_0\}| = |\{(v - k_1)^+ > k_1\}| \le \left(\frac{\|(v - k_1)^+\|_{L^p(K_r)}}{k_1}\right)^p$$

Thus, $v(x,t) \le C_1$ where C_1 depends only on the data. The same is true for $-v(x,t)$.

To verify the claim, we multiply equation 5.5 by $p[(v - k_1)^+]^{p-1}$ and integrate over B_r for some fixed t, $r > r_0$.

$$\int p(v - k_1)^{+p-1} v_t dx + p \int (\phi_n'(v) + \varepsilon)\nabla v \cdot \nabla((v - k_1)^{+p-1})dx$$

$$= \int pf_{0\varepsilon}(v - k_1)^{+p-1}dx - \int pf_{i\varepsilon}((v - k_1)^{+p-1})_{x_i} dx$$

Therefore

$$\frac{d}{dt}\int (v - k_1)^{+p}dx + p(p - 1)\int (\phi_n'(v) + \varepsilon)(v - k_1)^{+p-2}|\nabla v|^2 dx$$

$$\le p \int |f_{0\varepsilon}||(v - k_1)^{+p-1}|dx + p(p - 1)\int |f_{i\varepsilon}||(v - k_1)^{+p-2}||v_{x_i}|dx$$

We will use instead the estimate

$$|\{|f| > k\}| \leq \left(\frac{\|f\|_p}{k}\right)^p \tag{5.6}$$

which is valid for $f \in L^p(R_T)$.

<u>Proposition 5.1</u>. There exists a constant C_1 depending only on the data, such that

$$|v(x,t)| \leq C_1$$

if v is any solution of (5.5) (n,ε,r), $r > r_0$.

<u>Proof</u>. Apply Proposition 2.1 with $Q_T = K_r$ and $k_0 = \max(1, 2\|v_0\|_{L^\infty})$. We obtain the estimate

$$v(x,t) \leq C(\text{data}, \mu(k_0))$$

Set $k_1 = \frac{k_0}{2}$. We claim that

$$\|(v - k_1)^+\|_{L^p(K_r)} \leq C \tag{5.7}$$

where C depends only on the data. If this is done, then it follows from 5.6 that

$$\mu(k_0) = |\{v > k_0\}| = |\{(v - k_1)^+ > k_1\}| \leq \left(\frac{\|(v - k_1)^+\|_{L^p(K_r)}}{k_1}\right)^p$$

Thus, $v(x,t) \leq C_1$ where C_1 depends only on the data. The same is true for $-v(x,t)$.

To verify the claim, we multiply equation 5.5 by $p[(v - k_1)^+]^{p-1}$ and integrate over B_r for some fixed t, $r > r_0$.

$$\int p(v - k_1)^{+p-1} v_t\,dx + p\int (\phi_n'(v) + \varepsilon)\nabla v \cdot \nabla((v - k_1)^{+p-1})\,dx$$

$$= \int pf_{0\varepsilon}(v - k_1)^{+p-1}dx - \int pf_{i\varepsilon}((v - k_1)^{+p-1})_{x_i}dx$$

Therefore

$$\frac{d}{dt}\int (v - k_1)^{+p}dx + p(p-1)\int (\phi_n'(v) + \varepsilon)(v - k_1)^{+p-2}|\nabla v|^2 dx$$

$$\leq p\int |f_{0\varepsilon}||(v - k_1)^{+p-1}|dx + p(p-1)\int |f_{i\varepsilon}||(v - k_1)^{+p-2}||v_{x_i}|dx$$

$$\leq p \int |f_{0\varepsilon}||(v - k_1)^{+p-1}|dx + p(p-1)\varepsilon_1 \int |(v - k_1)^{+p-2}||\nabla v|^2 dx$$

$$+ \frac{p(p-1)}{4\varepsilon_1} \int f_\varepsilon^2 |(v - k_1)^{+p-2}|dx$$

Recall that $\phi_n'(v) + \varepsilon \geq \mu_2(k_1) > 0$ if $v > k_1$. Choose $\varepsilon_1 = \frac{\mu_2(k_1)}{2}$. Then

$$\frac{d}{dt} \int (v - k_1)^{+p} \leq p \int |f_{0\varepsilon}|(v - k_1)^{+p-1}dx + \frac{p(p-1)}{2\mu_2(k_1)} \int f_\varepsilon^2 (v - k_1)^{+p-2}dx$$

$$\leq \int |f_{0\varepsilon}|^p dx + (p-1) \int (v - k_1)^{+p}dx$$

$$+ \frac{p-1}{\mu_2(k_1)} \int |f_\varepsilon|^p dx + \frac{(p-1)(p-2)}{2\mu_2(k_1)} \int (v - k_1)^{+p}dx \ .$$

Set $g(t) = \int_{B(r)} (v(x,t) - k_1)^{+p}dx$. We have just shown

$$g'(t) - \alpha g(t) \leq h(t)$$

where $h(t) = \int |f_{0\varepsilon}(x,t)|^p dx + \frac{p-1}{\mu_2(k_1)} \int |f_\varepsilon(x,t)|^p dx$ and $\alpha = \max\left(p - 1, \frac{(p-1)(p-2)}{2\mu_2(k_1)}\right)$.

By hypothesis $g(0) = 0$. Hence, by Gronwall's inequality

$$g(t) \leq e^{\alpha t} \int_0^t e^{-\alpha s} h(s)ds \leq e^{\alpha T} \int_0^T h(s)ds$$

and so

$$\|(v - k_1)^+\|^p_{L^p(K_r)} = \int_0^T g(t)dt \leq Te^{\alpha T} \int_0^T h(s)ds$$

$$= Te^{\alpha T}\left(\|f_{0\varepsilon}\|_p^p + \frac{p-1}{\mu_2(k_1)} \|f_\varepsilon\|_p^p\right)$$

which is bounded by some constant depending only on the data.//

It now follows that the same estimate holds for solutions of 5.3 (n,ε), and that solutions of 5.4 (n,ε) are also uniformly bounded.

Next we derive local L^2 estimates for ∇u and ∇v.

$$\leq p \int |f_{0\varepsilon}||(v - k_1)^{+p-1}|dx + p(p - 1)\varepsilon_1 \int |(v - k_1)^{+p-2}||\nabla v|^2 dx$$

$$+ \frac{p(p - 1)}{4\varepsilon_1} \int f_\varepsilon^2 |(v - k_1)^{+p-2}|dx$$

Recall that $\phi_n'(v) + \varepsilon \geq \mu_2(k_1) > 0$ if $v \geq k_1$. Choose $\varepsilon_1 = \frac{\mu_2(k_1)}{2}$. Then

$$\frac{d}{dt} \int (v - k_1)^{+p} \leq p \int |f_{0\varepsilon}|(v - k_1)^{+p-1}dx + \frac{p(p - 1)}{2\mu_2(k_1)} \int f_\varepsilon^2 (v - k_1)^{+p-2}dx$$

$$\leq \int |f_{0\varepsilon}|^p dx + (p - 1) \int (v - k_1)^{+p} dx$$

$$+ \frac{p - 1}{\mu_2(k_1)} \int |f_\varepsilon|^p dx + \frac{(p - 1)(p - 2)}{2\mu_2(k_1)} \int (v - k_1)^{+p} dx .$$

Set $g(t) = \int_{B(r)} (v(x,t) - k_1)^{+p} dx$. We have just shown

$$g'(t) - \alpha g(t) \leq h(t)$$

where $h(t) = \int |f_{0\varepsilon}(x,t)|^p dx + \frac{p - 1}{\mu_2(k_1)} \int |f_\varepsilon(x,t)|^p dx$ and $\alpha = \max\left(p - 1, \frac{(p - 1)(p - 2)}{2\mu_2(k_1)}\right)$.

By hypothesis $g(0) = 0$. Hence, by Gronwall's inequality

$$g(t) \leq e^{\alpha t} \int_0^t e^{-\alpha s} h(s)ds \leq e^{\alpha T} \int_0^T h(s)ds$$

and so

$$\|(v - k_1)^+\|^p_{L^p(K_r)} = \int_0^T g(t)dt \leq Te^{\alpha T} \int_0^T h(s)ds$$

$$= Te^{\alpha T}\left(\|f_{0\varepsilon}\|_p^p + \frac{p - 1}{\mu_2(k_1)} \|f_\varepsilon\|_p^p\right)$$

which is bounded by some constant depending only on the data.//

It now follows that the same estimate holds for solutions of 5.3 (n,ε), and that solutions of 5.4 (n,ε) are also uniformly bounded.

Next we derive local L^2 estimates for ∇u and ∇v.

Proposition 5.2. Let v be a solution of 5.[illegible] and [illegible] (v). Then

$$\|\nabla v\|_{L^2(K_r)} \le C(\varepsilon, r, \text{data}) \qquad (5.8)$$

$$\|u\|_{L^2(K_r)} \le C(r, \text{data}) \qquad (5.9)$$

Proof. Let $\zeta(x)$ be a smooth test function, $0 \le \zeta \le 1$, $\zeta = 1$ on B_r with compact support in B_{r+1}, $|\nabla\zeta| \le 2$. Multiply 5.3 (n,ε) by $\zeta^2 v$ and integrate over K_{r+1}. This yields the inequality

$$\frac{\varepsilon}{4}\iint_{K_{r+1}} \zeta^2|\nabla v|^2 dxdt \le \frac{1}{2}\iint_{B_{r+1}} \zeta^2 v^2 [illegible] dx + [illegible] \iint_{K_{r+1}} v^2 dxdt$$

$$+ \frac{1}{p}\iint_{K_{r+1}} |f_n|^p dxdt + \frac{1}{p'}\iint_{K_{r+1}} |v|^{p'} dxdt$$

$$+ \frac{1}{2p}\iint_{K_{r+1}} |f_\varepsilon|^{2p} dxdt + \frac{1}{(2p)'}\iint_{K_{r+1}} [illegible] dxdt$$

$$+ \frac{2}{p}\iint_{K_{r+1}} |f_\varepsilon|^{2p} dxdt + \frac{4}{(2p)'}\iint_{K_{r+1}} |v|^{(2p)'} dxdt$$

where $p' = \dfrac{p}{p-1}$. Therefore

$$\iint_{K_r} |\nabla v|^2 dxdt \le \iint_{K_{r+1}} \zeta^2|\nabla v|^2 dxdt \le C(\varepsilon, r, \text{data})$$

The proof of 5.9 is similar; in this case the equation for u is multiplied by [illegible]. Recall that $[\beta_n(u)]_t u = [B_n(u)]_t$ where $B_n(x) = \int_0^x \beta_n'(s)s\,ds + x\beta_n(x)$. [illegible] one must estimate the integral

$$\iint_{K_{r+1}} (1 + \beta_n'(u))u^2 |\nabla\zeta|^2 dxdt$$

This causes no problem since the approximations β_n may be constructed so that $|s\beta_n'(s)| \le C$ independent of n, for bounded s.//

Theorem 2.1 may now be invoked to show the equicontinuity of solutions of 5.3 (n,ε) and 5.4 (n,ε) for fixed $\varepsilon > 0$. Thus, by Arzela-Ascoli theorem [illegible]

Proposition 5.2. Let v be a solution of 5.[illegible] (n,ϵ) and [illegible] $_n(v)$. Then

$$\|\nabla v\|_{L^2(K_r)} \leq C(\epsilon, r, \text{data}) \tag{5.8}$$

$$\|u\|_{L^2(K_r)} \leq C(\epsilon, \text{data}) \tag{5.9}$$

Proof. Let $\zeta(x)$ be a smooth test function, $0 \leq \zeta \leq 1$, $\zeta = 1$ on B_r with compact support in B_{r+1}, $|\nabla\zeta| \leq 2$. Multiply 5.3 (n,ϵ) by $\zeta^2 v$ and integrate over K_{r+1}. This yields the inequality

$$\frac{\epsilon}{4}\iint_{K_{r+1}} \zeta^2|\nabla v|^2 dx\,dt \leq \frac{1}{2}\iint_{K_{r+1}} \zeta^2 v^2 [illegible]\, dx + [illegible] \iint_{K_{r+1}} v^2 dx\,dt$$

$$+ \frac{1}{p}\iint_{K_{r+1}} |f_n|^p dx\,dt + \frac{1}{p'}\iint_{K_{r+1}} [illegible]^{p'} dx\,dt$$

$$+ \frac{1}{2p}\iint_{K_{r+1}} |f_\epsilon|^{2p} dx\,dt + \frac{1}{2p'}\iint_{K_{r+1}} \zeta^{p'} dx\,dt$$

$$+ \frac{2}{n}\iint_{K_{r+1}} |f_\epsilon|^{2p} dx\,dt + \frac{4}{(2p)'}\iint_{K_{r+1}} |v|^{(2p)'} dx\,dt$$

where $p' = \frac{p}{p-1}$. Therefore

$$\iint_{K_r} |\nabla v|^2 dx\,dt \leq \iint_{K_{r+1}} \zeta^2|\nabla v|^2 dx\,dt \leq C(\epsilon, r, \text{data})$$

The proof of 5.9 is similar; in this case the equation for u is multiplied by [illegible]. Recall that $[\beta_n(u)]_t u = [B_n(u)]_t$ where $B_n(x) = \int_0^x \beta_n'(s)s\,ds + x\beta_n(x)$. Also [illegible] one must estimate the integral

$$\iint_{K_{r+1}} (1 + s\beta_n'(u)u^2)|\nabla\zeta|^2 dx\,dt$$

This causes no problem since the approximations β_n may be constructed so that $|s\beta_n'(s)| \leq C$ independent of n, for bounded s.//

Theorem 2.1 may now be invoked to show the equicontinuity of solutions of 5.3 (n,ϵ) and 5.4 (n,ϵ) for fixed $\epsilon > 0$. Thus, by Arzela-Ascoli [illegible]

$$u^\varepsilon = \lim_{n_k \to \infty} {}^{n_k}u^\varepsilon$$

$$v^\varepsilon = \lim_{n_k \to \infty} {}^{n_k}v^\varepsilon$$

for some subsequence n_k, and the convergence is uniform on compact sets. By Proposition 5.2 $\{\nabla {}^{n_k}u^\varepsilon\}$, $\{\nabla {}^{n_k}v^\varepsilon\}$ are weakly compact on any fixed compact set in R_T. By a diagonal argument we may find functions $\bar{u}^\varepsilon$ and $\bar{v}^\varepsilon \in L^2_{loc}(R_T)$ such that

$$\nabla {}^{n_k}u^\varepsilon \to \bar{u}^\varepsilon$$

$$\nabla {}^{n_k}v^\varepsilon \to \bar{v}^\varepsilon$$

weakly in $L^2(K_r)$ for any r, and for some further subsequence which we again call n_k. It follows that $\bar{u}^\varepsilon = \nabla u^\varepsilon$, $\bar{v}^\varepsilon = \nabla v^\varepsilon$ in the sense of distributions, hence also in $L^2_{loc}(R_T)$. We thus obtain, analogously to 2.11

$$\iint_{R_T} (v^\varepsilon \psi_t - \nabla(u^\varepsilon + \varepsilon v^\varepsilon)\cdot\nabla\psi + f_{0\varepsilon}\psi - f_{i\varepsilon}\psi_{x_i})dxdt + \int_{\mathbb{R}^N} v_{0\varepsilon}(x)\psi(x,0)dx = 0 \tag{5.10}$$

for $\psi \in C_0^\infty(\mathbb{R}^N \times [0,T))$.

The local results of Sections 3 and 4 remain valid; the modulus of continuity of u^ε depends on the initial and boundary conditions and on $|\Omega|$ only through its dependence on the global L^∞ norm. In particular, the functions $\{u^\varepsilon\}$, $\{v^\varepsilon\}$ are equicontinuous. Thus we may define

$$u(x,t) = \lim_{\varepsilon_k \to 0} u^{\varepsilon_k}(x,t)$$

$$v(x,t) = \lim_{\varepsilon_k \to 0} v^{\varepsilon_k}(x,t)$$

where the convergence is uniform on compact sets. Clearly $v(x,t) = \beta(u(x,t))$ for

$$u^\varepsilon = \lim_{n_k \to \infty} {}^{n_k}u^\varepsilon$$

$$v^\varepsilon = \lim_{n_k \to \infty} {}^{n_k}v^\varepsilon$$

for some subsequence n_k, and the convergence is uniform on compact sets. By Proposition 5.2 $\{\nabla {}^{n_k}u^\varepsilon\}$, $\{\nabla {}^{n_k}v^\varepsilon\}$ are weakly compact on any fixed compact set in R_T. By a diagonal argument we may find functions $\bar{u}^\varepsilon$ and $\bar{v}^\varepsilon \in L^2_{loc}(R_T)$ such that

$$\nabla {}^{n_k}u^\varepsilon \to \bar{u}^\varepsilon$$

$$\nabla {}^{n_k}v^\varepsilon \to \bar{v}^\varepsilon$$

weakly in $L^2(K_r)$ for any r, and for some further subsequence which we again call n_k. It follows that $\bar{u}^\varepsilon = \nabla u^\varepsilon$, $\bar{v}^\varepsilon = \nabla v^\varepsilon$ in the sense of distributions, hence also in $L^2_{loc}(R_T)$. We thus obtain, analogously to 2.11

$$\iint_{R_T} (v^\varepsilon \psi_t - \nabla(u^\varepsilon + \varepsilon v^\varepsilon)\cdot\nabla\psi + f_{0\varepsilon}\psi - f_{i\varepsilon}\psi_{x_i})\,dxdt + \int_{\mathbb{R}^N} v_{0\varepsilon}(x)\psi(x,0)\,dx = 0 \tag{5.10}$$

for $\psi \in C_0^\infty(\mathbb{R}^N \times [0,T))$.

The local results of Sections 3 and 4 remain valid; the modulus of continuity of u^ε depends on the initial and boundary conditions and on $|\Omega|$ only through its dependence on the global L^∞ norm. In particular, the functions $\{u^\varepsilon\}$, $\{v^\varepsilon\}$ are equicontinuous. Thus we may define

$$u(x,t) = \lim_{\varepsilon_k \to 0} u^{\varepsilon_k}(x,t)$$

$$v(x,t) = \lim_{\varepsilon_k \to 0} v^{\varepsilon_k}(x,t)$$

where the convergence is uniform on compact sets. Clearly $v(x,t) = \beta(u(x,t))$ for

$(x,t) \in R_T$. As before we have $\nabla u \in L^2_{loc}(R_T)$ and $\nabla u^{\varepsilon_k} \to \nabla u$ weakly in L^2 on any compact set.

Thus we may pass to the limit in 5.10 through the sequence ε_k, which yields 5.2. Theorem 5.1 is therefore proved.

Remark. This result extends to arbitrary unbounded space domains Ω as long as there exists a sequence of subdomains $\Omega_n \subset\subset \Omega$ such that Ω_n has smooth boundary, $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. We specify zero boundary conditions on $\partial\Omega \times (0,T)$ as before.

SECTION 6.

Consider again the case of a bounded domain Ω. In place of the L^p conditions on F we may assume instead conditions involving the spaces $L^{q,r}(Q_T)$, where

$$L^{q,r}(Q_T) = \{f \text{ measurable on } Q_T : \|f\|_{q,r,Q_T} < \infty\}$$

$$\|f\|_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega |f(x,t)|^q dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}}$$

Analogously to the L^p case we define

$$G_{q,r}(Q_T) = \{F \in \mathcal{D}'(Q_T) : F = f_0 + (f_i)_{x_i},\ f_0, f_i^2 \in L^{q,r}(Q_T)\}$$

and then define $\hat{G}_{q,r}(Q_T)$ in the obvious way. The correct generalization of $p > \frac{N+2}{2}$ is $\frac{1}{r} + \frac{N}{2q} < 1$, $r \geq 1$, $q \geq 1$.

In this section we assume the hypotheses H1 and H3 and that

$$F \in \hat{G}_{q,r}(Q_T) \quad \text{for} \quad \frac{1}{r} + \frac{N}{2q} < 1 .$$

As before, in the nondegenerate case, it is known that the problem 1.1 has bounded continuous solutions for $F \in G_{q,r}(Q_T)$ with the same condition on the indices r and q.

For the proof we need a generalization of Lemma 2.1.

Lemma 6.1. There exists a constant C depending only on N such that

$$\|u\|_{q,r,Q_T} \leq C|u|_{V_2(Q_T)}$$

$(x,t) \in R_T$. As before we have $\nabla u \in L^2_{loc}(R_T)$ and $\nabla u^{\varepsilon_k} \to \nabla u$ weakly in L^2 on any compact set.

Thus we may pass to the limit in 5.10 through the sequence ε_k, which yields 5.2. Theorem 5.1 is therefore proved.

Remark. This result extends to arbitrary unbounded space domains Ω as long as there exists a sequence of subdomains $\Omega_n \subset\subset \Omega$ such that Ω_n has smooth boundary, $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. We specify zero boundary conditions on $\partial\Omega \times (0,T)$ as before.

SECTION 6.

Consider again the case of a bounded domain Ω. In place of the L^p conditions on F we may assume instead conditions involving the spaces $L^{q,r}(Q_T)$, where

$$L^{q,r}(Q_T) = \{f \text{ measurable on } Q_T : \|f\|_{q,r,Q_T} < \infty\}$$

$$\|f\|_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega |f(x,t)|^q dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}}$$

Analogously to the L^p case we define

$$G_{q,r}(Q_T) = \{F \in \mathcal{D}'(Q_T) : F = f_0 + (f_i)_{x_i},\ f_0, f_i^2 \in L^{q,r}(Q_T)\}$$

and then define $\hat{G}_{q,r}(Q_T)$ in the obvious way. The correct generalization of $p > \frac{N+2}{2}$ is $\frac{1}{r} + \frac{N}{2q} < 1$, $r > 1$, $q > 1$.

In this section we assume the hypotheses H1 and H3 and that

$$F \in \hat{G}_{q,r}(Q_T) \quad \text{for} \quad \frac{1}{r} + \frac{N}{2q} < 1 .$$

As before, in the nondegenerate case, it is known that the problem 1.1 has bounded continuous solutions for $F \in G_{q,r}(Q_T)$ with the same condition on the indices r and q.

For the proof we need a generalization of Lemma 2.1.

Lemma 6.1. There exists a constant C depending only on N such that

$$\|u\|_{q,r,Q_T} \le C|u|_{V_2(Q_T)}$$

for $u \in \overset{\circ}{V}_2(Q_T)$ provided q, r satisfy

$$\frac{1}{r} + \frac{N}{2q} = \frac{N}{4}$$

$$\begin{array}{lll} r \in [2,\infty] & q \in \left[2, \dfrac{2N}{N-2}\right] & N \geq 3 \\ r \in (2,\infty] & q \in [2,\infty) & N = 2 \\ r \in [4,\infty] & q \in [2,\infty] & N = 1 \end{array}$$

See pages 74-75 of [14].

Theorem 2.1 remains true with these modified conditions on f_0 and f_1.

The proof of interior continuity proceeds as in the original case; non-trivial modifications are necessary only in Propositions 2.1 and 3.1. We sketch here the required changes. Put

$$\theta = \frac{r'}{q'} \qquad r' = \frac{r}{r-1} \qquad q' = \frac{q}{q-1}$$

$$p = 1 + \frac{2}{N\theta}$$

$$\mu(k) = \int_0^T |A_k(t)|^{\theta} dt$$

The pair $(2p, 2p\theta)$ satisfy the requirements of Lemma 6.1, i.e.

$$\frac{1}{2p\theta} + \frac{N}{4p} = \frac{N}{4}$$

In Proposition 2.1 no change is necessary up to Equation 2.5. Then use Lemma 6.1 to derive

$$\|(v-k)^+\|^2_{2p,2p\theta} \leq C\Big(\int_0^T \int_{A_k(t)} f^2 dx\,dt$$

$$+ \Big(\int_0^T \Big(\int_{A_k(t)} |f_0|^{(2p)'} dx\Big)^{\frac{(2p\theta)'}{(2p)'}} dt\Big)^{\frac{2}{(2p\theta)'}}\Big)$$

where a prime denotes the usual Holder conjugate exponent. From this it follows that

$$\mu(h) \leq \frac{\mu(k)^{\beta}}{(h-k)^{\alpha}} \qquad h > k > k_0$$

for $u \in \overset{\circ}{V}_2(Q_T)$ provided q,r satisfy

$$\frac{1}{r} + \frac{N}{2q} = \frac{N}{4}$$

$$\begin{array}{lll} r \in [2,\infty] & q \in [2, \frac{2N}{N-2}] & N \geq 3 \\ r \in (2,\infty] & q \in [2,\infty) & N = 2 \\ r \in [4,\infty] & q \in [2,\infty] & N = 1 \end{array}$$

See pages 74-75 of [14].

Theorem 2.1 remains true with these modified conditions on f_0 and f_1.

The proof of interior continuity proceeds as in the original case; non-trivial modifications are necessary only in Propositions 2.1 and 3.1. We sketch here the required changes. Put

$$\theta = \frac{r'}{q'} \qquad r' = \frac{r}{r-1} \qquad q' = \frac{q}{q-1}$$

$$p = 1 + \frac{2}{N\theta}$$

$$\mu(k) = \int_0^T |A_k(t)|^{\theta} dt$$

The pair $(2p, 2p\theta)$ satisfy the requirements of Lemma 6.1, i.e.

$$\frac{1}{2p\theta} + \frac{N}{4p} = \frac{N}{4}$$

In Proposition 2.1 no change is necessary up to Equation 2.5. Then use Lemma 6.1 to derive

$$\|(v-k)^+\|^2_{2p,2p\theta} \leq C\Big(\int_0^T \int_{A_k(t)} f^2 dx dt$$

$$+ \Big(\int_0^T \Big(\int_{A_k(t)} |f_0|^{(2p)'} dx\Big)^{\frac{(2p\theta)'}{(2p)'}} dt\Big)^{\frac{2}{(2p\theta)'}}\Big)$$

where a prime denotes the usual Holder conjugate exponent. From this it follows that

$$\mu(h) \leq \frac{\mu(k)^{\beta}}{(h-k)^{\alpha}} \qquad h > k > k_0$$

for $\beta = \frac{p\theta}{r'}$, $\alpha = 2p\theta$; $\beta > 1$ since $\frac{1}{r} + \frac{N}{2q} < 1$, hence *Lemma* 2.3 applies as before. See [14], p. 181 for a slightly different proof in the non-degenerate case.

In Proposition 3.1 we may continue as before until line 3.8. Then again we use Lemma 6.1 to derive an estimate for $\|\zeta(w - k)^+\|_{2p,2p\theta}$.

Define

$$J_m = \frac{1}{|Q_0|}\left(\int_{Q_m}\left(\int (w - k_m)^{+2}dx\right)^{\theta}dt\right)^{\frac{1}{\theta}}$$

$$\mu(k,Q(R)) = \int_{t_0-R^2}^{t_0} |A_k(t)|^{\theta}dt$$

where

$$A_k(t) = \{x \in B(x_0,R) : w(x,t) > k\}$$

The inequalities

$$\mu(k_{m+1},Q_m) \le \left(\frac{|Q_0|J_m}{(k_{m+1} - k_m)^2}\right)^{\frac{1}{\theta}}$$

and

$$|Q_0|J_{m+1} \le \|(w - k_{m+1})^+\|^2_{2p,2p\theta,Q_{m+1}}\,\mu(k_{m+1},Q_m)^{\frac{1}{p'}}$$

are valid. From these we can again derive the inequality

$$J_{m+1} \le K_1 K_2^m J_m^{1+b_1}$$

with

$$b_1 = \frac{1}{q'} + \frac{1}{p'} - 1 > 0$$

The cases $\theta \ge 1$ and $\theta < 1$ must be considered spearately.

With these changes the rest of the proof proceeds as in section [illegible]

Theorem 1.1 remains valid under these conditions on F.

BIBLIOGRAPHY

[1] D. G. Aronson and P. Benilan, Regularité des solutions de l'équation des milieux porous dans $\mathbb{R}^N$, C. R. Acad. Sc. Paris 288 (1979), 103-105.

[2] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Nordhoff International Publishing Co., Leyden (1976).

[3] H. Brezis, Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, in "Contributions to Nonlinear Functional Analysis" (E. Zarantonello, ed.) Academic Press, New York, 1971.

[4] H. Brezis and W. Strauss, Semilinear elliptic equations in L^1, J. Math. Soc. Japan 25 (1973), 565-590.

[5] L. Caffarelli and L. C. Evans, Continuity of the temperature in the two phase Stefan problem, to appear.

[6] L. Caffarelli and A. Friedman, Continuity of the density of a gas flow in a porous medium, Trans. Am. Math. Soc. 252 (1979), 99-113.

[7] L. Caffarelli and A. Friedman, Regularity of the free boundary of a gas flow in an n-dimensional porous medium, Indiana Univ. J. of Math., Vol. 29 #3 (1980).

[8] M. Crandall and H. Brezis, Uniqueness of solutions of the initial value problem for $u_t - \Delta\phi(u) = 0$, J. Math. pures et appl. 58, 1979, p. 153-163.

[9] M. Crandall, H. Brezis, P. Benilan, A semilinear elliptic equation in $L^1(\mathbb{R}^N)$, Ann. Scuola Norm. Sup. Pisa, Serie IV, Vol. II, 1975, p. 523-555.

[10] M. Crandall, A. Pazy, P. Benilan, to appear.

[11] E. DiBenedetto, Continuity of weak solutions to certain singular parabolic equations, Technical Summary Report, Math. Research Center, Madison, WI.

[12] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, 1980.

[13] S. Kružkov, A priori estimates for generalized solutions of second order elliptic and parabolic equations, Soviet Math (1963), p. 757-761.

[14] O. A. Ladyzenskaia, V. A. Solonnikov, N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, 1968.

BIBLIOGRAPHY

[1] D. G. Aronson and P. Benilan, Regularité des solutions de l'équation des milieux porous dans $\mathbb{R}^N$, C. R. Acad. Sc. Paris 288 (1979), 103-105.

[2] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Nordhoff International Publishing Co., Leyden (1976).

[3] H. Brezis, Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, in "Contributions to Nonlinear Functional Analysis" (E. Zarantonello, ed.) Academic Press, New York, 1971.

[4] H. Brezis and W. Strauss, Semilinear elliptic equations in L^1, J. Math. Soc. Japan 25 (1973), 565-590.

[5] L. Caffarelli and L. C. Evans, Continuity of the temperature in the two phase Stefan problem, to appear.

[6] L. Caffarelli and A. Friedman, Continuity of the density of a gas flow in a porous medium, Trans. Am. Math. Soc. 252 (1979), 99-113.

[7] L. Caffarelli and A. Friedman, Regularity of the free boundary of a gas flow in an n-dimensional porous medium, Indiana Univ. J. of Math., Vol. 29 #3 (1980).

[8] M. Crandall and H. Brezis, Uniqueness of solutions of the initial value problem for $u_t - \Delta\phi(u) = 0$, J. Math. pures et appl. 58, 1979, p. 153-163.

[9] M. Crandall, H. Brezis, P. Benilan, A semilinear elliptic equation in $L^1(\mathbb{R}^N)$, Ann. Scuola Norm. Sup. Pisa, Serie IV, Vol. II, 1975, p. 523-555.

[10] M. Crandall, A. Pazy, P. Benilan, to appear.

[11] E. DiBenedetto, Continuity of weak solutions to certain singular parabolic equations, Technical Summary Report, Math. Research Center, Madison, WI.

[12] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, 1980.

[13] S. Kružkov, A priori estimates for generalized solutions of second order elliptic and parabolic equations, Soviet Math (1963), p. 757-761.

[14] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc., Providence, 1968.

[15] P. Sacks, Thesis, Univ. of Wisconsin, in preparation.

[16] E. Sabinina, On the Cauchy problem for the equation of nonstationary gas filtration in several space variables, Dokl. Akad. Nauk SSSR, 136 (1961), 1034-1037.

[17] A. Vol'pert and S. Hudjaev, Cauchy's problem for degenerate second order quasilinear parabolic equations, Math. U.S.S.R. Sbornik, 7, 1969, p. 365-387.

[18] E. DiBenedetto, Continuity of weak solutions to a general porous medium equation, to appear.

PS/ed

[15] P. Sacks, Thesis, Univ. of Wisconsin, in preparation.

[16] E. Sabinina, On the Cauchy problem for the equation of nonstationary gas filtration in several space variables, Dokl. Akad. Nauk SSSR, 136 (1961), 1034-1037.

[17] A. Vol'pert and S. Hudjaev, Cauchy's problem for degenerate second order quasilinear parabolic equations, Math. U.S.S.R. Sbornik, 7, 1969, p. 365-387.

[18] E. DiBenedetto, Continuity of weak solutions to a general porous medium equation, to appear.

PS/ed

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2214	2. GOVT ACCESSION NO. AD-100 617	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Paul Sacks		8. CONTRACT OR GRANT NUMBER(s) MCS80-02946 DAAG29-80-C-0041 MCS78-01245
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS (See Item 18 below)		12. REPORT DATE May 1981 13. NUMBER OF PAGES 43
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

singular parabolic equation, porous medium equation, regularity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A class of singular parabolic equations is studied. It is proved that an initial and boundary value problem associated with the equation has a bounded solution which is continuous in the interior of its domain of definition. The case of the inhomogeneous porous medium equation with bounded initial data is included.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2214	2. GOVT ACCESSION NO. AD-A100 617	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EXISTENCE AND REGULARITY OF SOLUTIONS OF INHOMOGENEOUS POROUS MEDIUM TYPE EQUATIONS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Paul Sacks		8. CONTRACT OR GRANT NUMBER(s) MCS80-02946 DAAG29-80-C-0041 MCS78-01245
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS (See Item 18 below)		12. REPORT DATE May 1981
		13. NUMBER OF PAGES 43
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

singular parabolic equation, porous medium equation, regularity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A class of singular parabolic equations is studied. It is proved that an initial and boundary value problem associated with the equation has a bounded solution which is continuous in the interior of its domain of definition. The case of the inhomogeneous porous medium equation with bounded initial data is included.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

DATE

END

DATE